PG Odd Semester (CBCS) Exam., December—2016

STATISTICS

(1st Semester)

Course No. : STS–103 (C)

(Probability Theory)

Full Marks : 75

Pass Marks : 30

Time : 3 hours

The figures in the margin indicate full marks for the questions

Answer **five** questions, taking **one** from each Unit

UNIT—I

1. *(a)* For a given sequence $\{A_n\}$, show that

(i) $\limsup A_n \quad \bigcap_{m\ \ 1} \ \bigcup_{n\ \ m} A_n$;

(ii) $\liminf A_n \quad \bigcup_{m\ \ 1} \ \bigcap_{n\ \ m} A_n$. 7

(b) If $\mathscr{H}$ is a field of subsets of , then show that—

(i) $\quad \mathscr{H}$;

(ii) $\quad \mathscr{H}$;

(iii) $E, F \quad \mathscr{H} \quad E \quad F \quad \mathscr{H}$. 1+1+2=4

(c) Given a measure space (, $\mathscr{B}$,); for every sequence $\{E_n\}$ in $\mathscr{B}$. Show that

$$\bigcup_{n\ \ 1} E_n \qquad\qquad (E_n)$$
$$_{n\ \ 1}$$ 4

2. *(a)* Given a measure space (, $\mathscr{B}$,), where is finite, for every sequence $\{E_n\}$ in $\mathscr{B}$. Show that—

(i) $\limsup \ (E_n) \quad (\limsup E_n)$;

(ii) $\liminf \ (E_n) \quad (\liminf E_n)$. 5

(b) State and prove the Hahn decomposition theorem. 5

(c) Define Lebesgue measure. How can probability be considered as a special type of measure? 5

UNIT—II

3. *(a)* Define probability space with examples. State and prove the total probability theorem. 2+4=6 *(b)* What is conditional probability? Consider a family with two children. Assuming that each child is likely to be a boy or a girl, what is the conditional probability that both child will be a boy? Given that—

(i) the elder child is a boy;

(ii) at least one of the child is a boy. 2+3=5

(c) Let the events $A_1, \cdots, A_n$ be independent and $P[A_k] \quad p_k$. Find the probability P that none of the events occur. Further show that $P \quad e^{\quad p_k}$. 4

4. *(a)* Prove that

$$P[E_i / A] \quad \frac{P[E_i] P[A / E_i]}{\overset{n}{\underset{i \quad 1}{}} P[E_i] P[A / E_i]}$$ 6

(b) Urn I contains 3 red, 4 black and urn II contains 2 red, 6 black balls. An urn is selected at random and a ball is drawn from it at random. If it is red, find the probability that the next ball drawn without replacement will also be red. 6

(c) Define mutually independent events. 3

UNIT—III

5. *(a)* Define random variables and types of random variables with examples. 5

(b) Show that for any random variable X

$$E \; \frac{1}{X} \qquad \frac{1}{E(X)}$$ 5

(c) Prove that—

(i) $P(a \quad X \quad b) \quad F(b) \quad F(a)$;

(ii) $F(\quad) \quad 0$ and $F(\quad) \quad 1$;

(iii) $\lim_{t \quad a} F(t) \quad F(a)$. 1+2+2=5

6. *(a)* The joint p.d.f. of X and Y is given by

$$f(x, y) \quad e^{(x \quad y)}, \quad x \quad 0, y \quad 0$$
$$0, \qquad \text{elsewhere}$$

Find the distribution of the following : 3+3=6

(i) $X \quad Y$

(ii) X / Y

(b) Let

$$f(x, y) \quad ky^2, \quad 0 \quad 2x \quad y \quad 2$$
$$0, \qquad \text{elsewhere}$$

Find k, the conditional mean and variance of X given Y. 3+3+3=9

UNIT—IV

7. *(a)* Using an important property of probability generating function (p.g.f.), prove that binomial distribution tends to Poisson distribution. 7

(b) A distribution is defined as

$$f(x) \quad \frac{n}{\lfloor n \quad 1} x^{n \quad 1} e^{\quad x}, \; x \quad 0$$

Find the characteristic function of $f(x)$ and hence deduce the mean and variance of $2X \quad 3$. 5

(c) Show that $M_{XY}(t_1, t_2) \quad M_X(t_1) \; M_Y(t_2)$ only if X and Y are independent of each other. 3

8. *(a)* State and prove the inversion theorem of characteristic function. 10

(b) Prove that if $\quad (t)$ is the characteristic function of a variate X, then the density of X is

$$f(x) \quad \frac{1}{2} \quad e^{\quad itx} \quad (t)\, dt$$ 5

UNIT—V

9. *(a)* State and prove Chebyshev's inequality. Prove that

$$P[X \quad a] \quad \frac{E(X)}{a}$$ 1+5+2=8

(b) State and prove Lindeberg-Levy central-limit theorem. 2+5=7

10. *(a)* State the Lindeberg-Feller central-limit theorem. 3

(b) Examine if the law of large numbers holds for the sequence of independent random variables $\{X_n\}$ with the distribution of X_n given by

$$f_n(x) \quad \begin{matrix} \frac{1}{|x|^3}, & |x| \quad 1 \\ 0, & \text{otherwise} \end{matrix}$$ 6

(c) Let $X_1, X_2, \cdots$ be i.i.d. variates with p.m.f.

$$f(x) \quad {}^{x}/(1 \quad), \quad x \quad 0, 1, 2, \cdots$$
$$0 \quad\quad 1$$

Show that weak law of large numbers (WLLN) is followed by X_i's. 6

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