2016/ODD/08/23/MTM–101/470

PG Odd Semester (CBCS) Exam., December—2016

MATHEMATICS

(1st Semester)

Course No. : MTMCC–101

(Algebra—I)

Full Marks : 70

Pass Marks : 28

Time : 3 hours

The figures in the margin indicate full marks for the questions

Answer **one** question from each Unit

UNIT—I

1. *(a)* Check whether the function $f : \mathbb{R} \to (0, 1)$ defined by

$$f(x) = \frac{|x|}{1 + |x|}$$

for all $x \in \mathbb{R}$ is bijective. 3

(b) Let G be a group such that order of each element of G is atmost 2. Show that G is Abelian. 3

(c) Prove that

$$H = \left\{ \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} : n \in \mathbb{Z} \right\}$$

is a cyclic subgroup of the group of all 2×2 invertible real matrices $GL(2, \mathbb{R})$. 5

(d) Let G be a finite cyclic group of order 10 and let a be a generator of G. Find the order of the subgroup of G generated by a^6. 3

2. *(a)* Let X and Y be non-empty sets and let $f : X \to Y$. Show that f is one-one iff $f^{-1}(f(A)) = A$ for all $A \subseteq X$. 4

(b) Show that

$$\{m \in \mathbb{N} : m < n \text{ and } (m, n) = 1\}$$

is a group with respect to multiplication modulo n, i.e.

$$a \otimes_n b = \begin{cases} ab, & \text{if } ab < n \\ r, & \text{if } ab \geq n \end{cases}$$

where by division algorithm $ab = nq + r$, $0 \leq r < n$ and $q \in \mathbb{Z}$. Here (m, n) stands for g.c.d. of m and n. 5

(c) Show that, every cyclic group is Abelian. Is the converse true? Justify. 3+2=5

UNIT—II

3. *(a)* Find the number of distinct 4 cycles in the group S_6. Does A_6 contain any 4 cycles? Justify. 2+2=4

(b) Find the order of the following permutations : 2+2=4

(i) $\begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 4 & 1 & 2 & 3 & 6 \end{matrix}$

(ii) $(1\ 2)\ (1\ 3)\ (4\ 5)\ (1\ 6)$

(c) Show that A_4 cannot have a subgroup of order 6. 3

(d) Find the index of the subgroup

$$(3\mathbb{Z},\quad)\text{ in }(\mathbb{Z},\quad)$$ 3

4. *(a)* Let $\quad (1\ 2\ 3)(1\ 4\ 5)$. Express $\quad^{99}$ in disjoint cycle form. 3

(b) Find the number of elements of order 6 in S_5. 3

(c) Let $H\ \{\quad S_5 : \ (1)\ \ 1$ and $\ (3)\ \ 3\}$. Show that H is a subgroup of G. Also, find the order of $|H|$. 2+2=4

(d) If $a\quad\mathbb{Z}$ and p is prime, then show that $a^p\quad a(\text{mod } p)$. 4

UNIT—III

5. *(a)* Let G be a group and let $H\quad G$. If K is a subgroup of G such that $K\quad H$, then prove or disprove that $K\quad G$. Here $A\quad B$ stands for A is a normal subgroup of B. 4

(b) Find the number of group homomorphisms from $\mathbb{Z}_4$ to $\mathbb{Z}_6$. 4

(c) Does there exist an epimorphism from a group of order 10 to a group of order 8? Justify. 2

(d) Prove or disprove the following statements : 2+2=4

(i) $(\mathbb{R},\quad)$ and $(\mathbb{Q},\quad)$ are isomorphic

(ii) $(\mathbb{R}^x, \cdot)$ and $(\mathbb{C}^x, \cdot)$ are isomorphic.

6. *(a)* Define normaliser of a subset of a group G. Show that if H is a subgroup of G, then H is a normal subgroup of the normaliser of H. 1+3=4

(b) Prove or disprove the following statements : 2+2=4

(i) If every subgroup of a group is normal, then the group must be Abelian.

(ii) If H is an Abelian normal subgroup of G such that G/H is Abelian, then G is Abelian.

(c) Define a homomorphism $f : S_4 \quad \{ \; 1, 1\}$ such that kernal of f is A_4 (with proper justification). 3

(d) Let $f : G \quad G$ be a homomorphism, where G and G are groups. If $a \quad G$, then show that $O(f(a)) | O(a)$. 3

UNIT—IV

7. *(a)* Let R be the ring of all real-valued functions defined on $\mathbb{R}$ under componentwise addition and multiplication.

(i) Find all zero divisors in R.

(ii) Determine all nilpotent elements of R.

(iii) Show that any non-zero element of R is either a zero divisor or a unit.

2+2+2=6

(b) Does there exist a finite field with characteristics O? Give example of an infinite field whose characteristic is 5. Justify your steps. 2+2=4

(c) Let F be a finite field with 3 elements. Show that the sum of the elements of F is 0, the zero element of F. 4

8. *(a)* Let F be a field with ch$F \quad 2$. If the non-zero elements of F form a cyclic group, then show that, F must be finite, where ch$F$ stands for characteristics of F. 4

(b) Let R and S be any two rings, at least one of which possesses unity. Show that any ideal of $R \quad S$ must be of the form $I \quad J$, where I is an ideal of R and J is an ideal of S. Hence or otherwise, find all the maximal ideals of $\mathbb{Z}_8 \quad \mathbb{Z}_{30}$. 3+2=5

(c) Let R be a commutative ring with unity and let I be a proper ideal of R with the property that every element of R that is not in I is a unit in R. Show that I is the unique maximal ideal of R. 5

UNIT—V

9. *(a)* Let R be a Euclidean domain. Show that every ideal of R is a principal ideal. 4

(b) Let F be a field and R be an integral domain in $F[x]$ generated by x^2 and x^3. Show that R is not a UFD. 4

(c) Determine all ring homomorphisms from—

(i) $\mathbb{Z}_{20}$ to $\mathbb{Z}_{30}$;

(ii) $\mathbb{Z} \quad \mathbb{Z}$ to $\mathbb{Z}$. 3+3=6

10. *(a)* In a PID, show that every non-zero prime ideal is maximal. 4

(b) Is there any ring homomorphism from the field $\mathbb{R}$ to some ring whose kernel is $\mathbb{Z}$? Justify. 3

(c) Let $m \quad \mathbb{N}$ and let $n \quad \mathbb{Z}$ be obtained from m by rearranging its digits in some random way. Show that $m \quad n$ is divisible by 9. 3

(d) Give an example, with justification, of an irreducible element in an integral domain, which is not a prime element. 4

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