PG Odd Semester (CBCS) Exam., December—2016

MATHEMATICS

(1st Semester)

Course No. : MTMCC–102

(Real Analysis—I)

Full Marks : 70
Pass Marks : 28

Time : 3 hours

The figures in the margin indicate full marks for the questions

Answer **one** question from each Unit

UNIT—I

1. *(a)* Show that the set of all real sequences with 0 and 1 only is uncountable. 4

(b) Let A be a bounded subset of $\mathbb{R}$ and let $a, b \in \mathbb{R}$ such that $a = \sup A$ and $b = \inf A$. Show that there exist sequences (a_n) and (b_n) in A such that

$$\lim_n a_n = a \text{ and } \lim_n b_n = b$$ 4

(c) Find the limit superior of the following sequences (x_n) (any *one*) , where

(i) $x_n = \frac{1}{n} + (-1)^{n-1}\frac{1}{n}, \quad n \in \mathbb{N}$

(ii) $x_n = 1 + \frac{(-1)^{n-1}}{n}, \quad n \in \mathbb{N}$ 3

(d) Test the convergence of the following series (any *one*) : 3

(i) $\sum_{n=1} \frac{1}{\log n}$

(ii) $\sum_{n=1} \frac{n}{2n-1}$

2. *(a)* Find the supremum and infimum of the set

$$\left\{\frac{1}{n} + \frac{1}{m} : n, m \in \mathbb{N}\right\}$$ 4

(b) Let (x_n) be a sequence in $\mathbb{R}$ such that

$$|x_n - x_{n-1}| \leq \frac{3}{2^n}, \quad n \in \mathbb{N}$$

Show that (x_n) is convergent. 4

(c) Test the convergence of the following sequences (x_n) (any *one*), where

(i) $x_n \quad \frac{n}{n^3 \quad 1} \quad \frac{2n}{n^3 \quad 2} \quad \cdots \quad \frac{n^2}{n^3 \quad n} \quad n \quad \mathbb{N}$

(ii) $x_1 \quad 4$ and $x_{n \quad 1} \quad 3 \quad \frac{2}{x_n} \quad n \quad \mathbb{N}$ 3

(d) Test the convergence of the following series (any *one*) : 3

(i) $\sum_{n \quad 1} (\quad 1)^{n \quad 1} \frac{1}{n(n \quad 1)}$

(ii) $\sum_{n \quad 1} ((n^3 \quad 1)^{1/3} \quad n)$

UNIT—II

3. (a) Show that a subset A of $\mathbb{R}$ is closed iff A contains all its limit points. 5

(b) Let $f : [a, b] \quad \mathbb{R}$ be a continuous function such that $f(a) \quad 0$ and $f(b) \quad 0$. Show that there exists $c \quad (a, b)$ such that $f(c) \quad 0$. 5

(c) Show that the function $f : (0, 1) \quad \mathbb{R}$ defined by

$$f(x) \quad e^x \cos\frac{1}{x} \quad x \quad (0, 1)$$

is not uniformly continuous. 4

4. (a) Criticise the following proof : 3

Let $\{A_k : k \quad \mathbb{N}\}$ be a collection of open subsets of $\mathbb{R}$. Let

$$\bigcap_{k \quad 1} A_k$$

Then there exists $x \quad \mathbb{R}$ such that $x \quad A_k$ for all $k \quad \mathbb{N}$. Thus there exists $r_k \quad 0$ such that $(x \quad r_k, x \quad r_k) \quad A_k$ for all $k \quad \mathbb{N}$. Let $r \quad \min\{r_k : k \quad \mathbb{N}\}$. Then $r \quad 0$ and

$$(x \quad r, x \quad r) \quad (x \quad r_k, x \quad r_k) \quad A_k$$

for all $k \quad \mathbb{N}$. This implies

$$(x \quad r, x \quad r) \quad \bigcap_{k \quad 1} A_k$$

and so $\bigcap_{k \quad 1} A_k$ is an open set.

(b) Let I be an interval in $\mathbb{R}$ and let $a, b(\quad a) \quad I$. Let $f : I \quad \mathbb{R}$ be a continuous function such that

$$\lim_{x \quad a} f(x) \qquad \text{and} \quad \lim_{x \quad b} f(x)$$

Show that there exists $c \quad (a, b)$ such that $f(c) \quad 0$. 5

(c) Show that a monotone function cannot have a discontinuity of second kind. Using the above fact, show that $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ defined by

$$f(x) = \frac{1}{x} \text{ for all } x \in \mathbb{R} \setminus \{0\}$$

is not a monotone function. 4+2=6

UNIT—III

5. *(a)* Examine whether $f : \mathbb{R} \to \mathbb{R}$ defined as below is differentiable at 0 (any *one*) : 5

(i) $$f(x) = \begin{cases} \frac{1}{2^{n-1}}, & \text{if } x = \frac{1}{2^n} \text{ for some } n \in \mathbb{N} \\ 0, & \text{otherwise} \end{cases}$$

(ii) $$f(x) = \begin{cases} x^2, & \text{if } x \in \mathbb{Q} \\ 0, & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

(b) Let $f : [0, 1] \to \mathbb{R}$ be a continuous function such that f is differentiable in $(0, 1)$. Show that there exists $x_0 \in (0, 1)$ such that $f'(x_0) = f(x_0)$. 5

(c) Show that the Maclaurin expansion for $\sin x$ converges to $\sin x$. 4

6. *(a)* Let

$$f(x) = \begin{cases} \frac{\sin x}{x}, & \text{if } x \neq 0, x \in \mathbb{R} \\ 1, & \text{if } x = 0 \end{cases}$$

Examine whether f is continuously differentiable. 5

(b) Find the number of distinct real roots of any *one* of the following equations : 5

(i) $e^{2x} - \cos x - x = 0$

(ii) $3^x + 4^x = 5^x$

(c) Examine whether 0 is a local maximum or a local minimum for the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \sin x - \frac{1}{6}x^3 \text{ for all } x \in \mathbb{R}$$ 4

UNIT—IV

7. *(a)* Show that a bounded function $f : [a, b] \to \mathbb{R}$ is Riemann integrable iff for each $\varepsilon > 0$, there exists a partition p of $[a, b]$ such that $U(f, p) - L(f, p) < \varepsilon$. 5

(b) Show that the sequence of functions (f_n) converges pointwise to $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 0$ for all $x \in \mathbb{R}$, where

$$f_n(x) = \frac{x}{n} \text{ for all } x \in \mathbb{R} \text{ and for all } n \in \mathbb{N}$$

Is the convergence uniform? Justify. 4+2=6

(c) Show that the series of functions

$$\sum_{n=1}^{\infty} f_n$$

is uniformly convergent, where for each $n \in \mathbb{N}$

$$f_n(x) = \frac{\sin x}{n^2} \text{ for all } x \in \mathbb{R}$$

3

8. *(a)* Show that a bounded function $f : [a, b] \to \mathbb{R}$ is Riemann integrable and

$$\int_a^b f(x)\,dx$$

if and only if there exists a sequence of partitions (p_n) of $[a, b]$ such that

$$\lim_n L(f, p_n) = \lim_n U(f, p_n)$$

4

(b) Let $f_n(x) = nx(1 - x^2)^n$ for all $x \in [0, 1]$. Find the pointwise limit f of (f_n). Does $\int_0^1 f_n(x)\,dx$ converge to $\int_0^1 f(x)\,dx$?

Justify. 4+2=6

(c) Show that

$$\sum_{n=1}^{\infty} x^n(1 - x^n)$$

converges pointwise but not uniformly on [0, 1]. 4

UNIT—V

9. *(a)* Find the radius of convergence of any *one* of the following power series : 3

(i) $\sum_{n=1}^{\infty} \frac{2^n}{\lfloor n} x^n$

(ii) $\sum_{n=1}^{\infty} [2 - (-1)^n]^n x^n$

(b) Using definition only, examine whether the following functions are of bounded variation (any *one*) : 5

(i) $f : [0, 1] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1, & \text{if } x \in [0, 1] \cap \mathbb{Q} \\ 0, & \text{if } x \in [0, 1] \cap (\mathbb{R} \setminus \mathbb{Q}) \end{cases}$$

(ii) $f : [0, 2\pi] \to \mathbb{R}$ defined by $f(x) = \sin x$ for all $x \in [0, 2\pi]$

(c) Prove that every Lipschitz continuous function is of bounded variation. Is the converse true? Justify. 3+3=6

10. *(a)* Let $f : [a, b]$ $\mathbb{R}$ be a function of bounded variation and let $g : [a, b]$ $\mathbb{R}$ be a bounded function on $[a, b]$ such that $\{x \quad [a, b] : f(x) \quad g(x)\}$ is finite. Show that g is a function of bounded variation. Can the finiteness of

$$\{x \quad [a, b] : f(x) \quad g(x)\}$$

be replaced by countability so that the conclusion holds? Justify. 4+2=6

(b) Find the total variation of the function $f : [0, 2]$ $\mathbb{R}$ defined by

$f(x)$ $x(x \quad 1)$ for all x $[0, 2]$ 4

(c) Define an absolutely continuous function. Give an example of an absolutely continuous function with proper justification. 1+3=4

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