PG Odd Semester (CBCS) Exam., December—2016

MATHEMATICS

(3rd Semester)

Course No. : MTMCC–301

(Real Analysis—II)

Full Marks : 70
Pass Marks : 28

Time : 3 hours

The figures in the margin indicate full marks for the questions

Answer **one** question from each Unit

UNIT—1

1. *(a)* Let $A \subseteq \mathbb{R}^n$ be such that every continuous function $f : A \to \mathbb{R}$ is bounded. Show that, A must be a compact subset of $\mathbb{R}^n$. 5

(b) Let $f : \mathbb{R}^2 \to \mathbb{R}^3$ be differentiable at $(0, 0)$. If

$$\lim_{r \to 0} \frac{f(r, r) - f(0, 0)}{r} = (2, 1, 1)$$

and $$\lim_{r \to 0} \frac{f(r, r) - f(0, 0)}{r} = (8, 13, 1)$$

then find, $\lim_{r \to 0} \frac{f(3r, 2r) - f(0, 0)}{r}$. 4

(c) Find the 3rd order Taylor polynomial of $f(x, y) = e^x \cos y$ about $(0, 0)$. 5

2. *(a)* If $f : \mathbb{R}^n \to \mathbb{R}^m$ be differentiable at $\underline{a}$ show that, f must be continuous at $\underline{a}$. Show further that, the directional derivatives of f exist at $\underline{a}$ along every direction $\underline{0} \neq \underline{u} \in \mathbb{R}^n$ and $f_{\underline{u}}(\underline{a}) = Df(\underline{a})(\underline{u})$. 5

(b) Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. Define $F : \mathbb{R}^2 \to \mathbb{R}$ by

$$F(x, y) = \int_0^{xy} f(t)\, dt, \quad (x, y) \in \mathbb{R}^2$$

Show that, F is differentiable on $\mathbb{R}^2$ and find $F(x, y)$ at any $(x, y) \in \mathbb{R}^2$. 4

(c) Let

$$f(x, y) = \begin{cases} x^2 \tan^{-1}\frac{y}{x} - y^2 \tan^{-1}\frac{x}{y} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

Examine the equality of

$$f_{xy}(0, 0) \quad f_{yx}(0, 0)$$

Are the conditions of Young's theorem satisfied by f? 5

UNIT—2

3. (a) Let B be the closed unit ball in $\mathbb{R}^n$ and $f : B \to B$ is continuous on B and is differentiable in B^0. If f vanishes on the boundary of B, show that, there exist $\underline{x}_0 \in B^0$ such that $Df(\underline{x}_0) = 0$. 5

(b) Find the points in $\mathbb{R}^2$ where $f : \mathbb{R}^2 \to \mathbb{R}^2$.

$$f(x, y) = (\cos x - \cos y, \sin x - \sin y)$$

is locally invertible. 5

(c) Find a set of sufficient conditions on the function $f : \mathbb{R}^2 \to \mathbb{R}$ which allows you to solve the equation $F(F(x, y), y) = 0$ for y as a function of x near $(0, 0)$. 4

4. (a) Find the distance between the ellipse $x^2 + 2y^2 = 1$ and the line $x + y = 4$ using Lagrange's multiplier method. 5

(b) State inverse function theorem. Show that, if $\mathbb{R}^n$ is open and if $f : \mathbb{R}^n$ is such that $f \in C(\ )$ and $\det [Df(\underline{x})] \neq 0$ for any $\underline{x}$, then f must be an open mapping. Is f necessarily a closed mapping? Justify your answer. 1+2+2=5

(c) Show that in a neighbourhood of any $(x_0, y_0, u_0, v_0) \in \mathbb{R}^4$ which satisfy the equations $x - e^u \cos v = 0$ and $v - e^y \sin x = 0$, there exist unique solution of the form $(u, v) = (x, y)$ satisfying

$$\det[\ (x, y)] \quad \frac{v}{x}$$ 4

UNIT—3

5. (a) Show that, every countable subset of $\mathbb{R}$ is Lebesgue measurable. 4

(b) Show that, if E is a set having positive Lebesgue outer measure, then E must have a bounded subset whose outer measure is positive. 3

(c) Let E_1 and E_2 are measurable subsets of $\mathbb{R}$. Show that,

$m(E_1 \quad E_2) \quad m(E_1 \quad E_2) \quad m(E_1) \quad m(E_2)$ 3

(d) Define $f : [0, 2) \quad R$ by

$$f(x) \quad \begin{matrix} x \ , & \text{if} \quad 0 \quad x \quad 1 \\ 3 \quad x \ , & \text{if} \quad 1 \quad x \quad 2 \end{matrix}$$

Find $m\ (A)$ where $A \quad f^{\ 1}\left(\frac{16}{9}, \frac{5}{4}\right)$. 4

6. (a) Show that, the collection of all Lebesgue measurable sets of $\mathbb{R}$ form an algebra of sets. 4

(b) Show that, for any bounded set E, there exist a G -set G such that $E \quad G$ and $m\ (G) \quad m\ (E)$. 4

(c) Show that, continuity of measure together with finite additivity of measure implies countable additivity. 3

(d) Does there exist a non-empty open set whose measure is 0? Also give an example of a non-empty closed set whose measure is 0. 2+1=3

UNIT—4

7. (a) If $f : E \quad \mathbb{R}$ is measurable and $f \quad g$ a.e. on E show that, $g : E \quad \mathbb{R}$ is also measurable. 3

(b) Let f_n be a sequence of measurable function on E such that $f_n \quad f$ a.e. on E. Show that f must be Lebesgue measurable. 4

(c) State and prove Bounded Convergence theorem. 5

(d) If is simple function on E and $A \quad E$, show that

$$A \quad E \qquad A \qquad 2$$

8. (a) Let f and g are measurable functions on E and $F : \mathbb{R}^2 \quad \mathbb{R}$ be continuous. Show that, $h : E \quad \mathbb{R}$ defined by

$$h(x) \quad F(f(x), g(x)) \qquad x \quad E$$

is measurable. Deduce that $f \quad g$ and fg are measurable. 4+2=6

(b) If and are simple function over a measurable set E of finite measure, show that

$$E \qquad E \quad E \qquad 4$$

(c) Let f be a bounded measurable function on a set of finite measure E. Show that, f must be Lebesgue integrable over E. 4

UNIT—5

9. (a) Let f be a non-negative measurable function on a measurable set E. Show that $\int_E f = 0$ iff $f = 0$ a.e. on E. 5

(b) State and prove Lebesgue Dominated Convergence theorem. 5

(c) For each $x \in [0, 1]$, define

$$f(x) = \begin{cases} x^2 & \text{if } x = \frac{1}{2^n} \text{ for some } n \in \mathbb{N} \\ x^3 & \text{if } x = \frac{1}{3^n} \text{ for some } n \in \mathbb{N} \\ x^4 & \text{otherwise} \end{cases}$$

Find $\int_{[0, 1]} f$. 4

10. (a) State Monotone Convergence theorem. Use it to show that, if f_n is a sequence of non-negative measurable function on E, then

$$\int_E \liminf_n f_n \leq \liminf_n \int_E f_n \qquad 1+3=4$$

(b) Let $f : E \to \mathbb{R}$ be a measurable function such that $|f|$ is Lebesgue integrable over E. Show that, f must be Lebesgue integrable over E. Is the result remain in case of Riemann integral? Justify. 3+2=5

(c) Let f be integrable over E and $\{E_n\}$ be a disjoint sequence of measurable subsets of E such that

$$E = \bigcup_{n=1}^{\infty} E_n$$

Show that

$$\int_E f_n \to \int_E f \qquad 5$$

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