PG Odd Semester (CBCS) Exam., December—2016

MATHEMATICS

(3rd Semester)

Course No. : MTMCC–302

(Discrete Mathematics)

Full Marks : 70
Pass Marks : 28

Time : 3 hours

The figures in the margin indicate full marks for the questions

Answer **five** questions, taking **one** from each Unit

UNIT—I

1. *(a)* Let D_n be the set of all derangements of $\{1, 2, \cdots, n\}$. Prove that

$$|D_n| = \sum_{i=0}^{n} (-1)^i \binom{n}{i} (n-i)! \qquad 5$$

(b) Determine all solutions in integers of the Diophantine equation $56x + 72y = 40$. 4

(c) If $p > 5$ is a prime, show that $p^2 - 2$ is a composite. 3

(d) Show that there is no solutions in integers x, y satisfying $4x^2 - y^2 = 102$. 2

2. *(a)* For a subset X of $\{1, 2, \cdots, 9\}$, define

$$\sigma(X) = \sum_{x \in X} x$$

Prove that among any 26 subsets of $\{1, 2, \cdots, 9\}$ each having size at most three, there are subsets A and B such that $\sigma(A) = \sigma(B)$. 5

(b) For $n > 1$, $n \in \mathbb{N}$, can

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$$

be an integer? Justify your answer in brief. 5

(c) $n > 1$, $n \in \mathbb{N}$, prove that there are n consecutive composite integers. 4

UNIT—II

3. *(a)* Prove that for an odd prime p, $p \equiv 1 \bmod 4$ if and only if $-1 \equiv x^2 \bmod p$ for some $x \in \mathbb{N}$. 6

(b) How many zeros are there at the end of $|300|$? Justify your answer in brief. 4

(c) Find the remainder when 4444^{4444} is divided by 9. 4

4. (a) Prove that if $n \quad 1$ is a prime if and only if $(n \quad 1)! \quad 1 \bmod n$. 5

(b) Find the last two digits of $7^{7^{1000}}$. 5

(c) If p, q are distinct primes, prove that

$$pq \mid a^{pq} \quad a^p \quad a^q \quad a$$

for any $a \quad \mathbb{Z}$. 4

UNIT—III

5. (a) Let f be a multiplicative function and let

$$F(n) \quad \sum_{d \mid n} f(d)$$

Prove that F is multiplicative. 5

(b) Let $a, n \quad \mathbb{N}$, $a \quad 1$. Prove that—

(i) $n \mid (a^n \quad 1)$;

(ii) $\sum_{d \mid n} n^{(n)/2} \quad (n)$;

(iii) $\sum_{k \quad 1}^{n} (k) \quad \sum_{j \quad 1}^{n} \frac{n}{j}$. 3+3+3=9

6. (a) If p and q are distinct odd primes, prove that

$$\frac{p}{q} \quad \frac{q}{p} \quad (\quad 1)^{\frac{p \quad 1}{2} \frac{q \quad 1}{2}}$$ 6

(b) If p and $2p \quad 1$ are both odd primes, prove that the integers $(\quad 1)^{(p \quad 1)/2} 2$ is a primitive root of $2p \quad 1$. 5

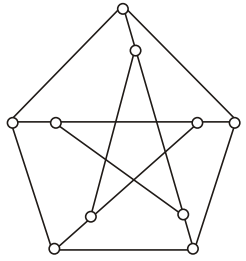
(c) Prove that

$$\sum_{d \mid n} | \quad (d)| \quad 2^{\quad (n)}$$ 3

UNIT—IV

7. (a) Prove that a graph is bipartite if and only if it contains no cycle of odd length. 6

(b) Find cycles of length 5, 6, 8 and 9 in the graph given below : 4



(c) If G is a graph which contains a cycle, prove that $g(G) \quad 2D(G) \quad 1$. 4

8. *(a)* Prove that a connected graph G with at least two edges is a block if and only if any two adjacent edges lie in a cycle. 5

(b) For any graph G, prove that—

(i) $R(G)$ $D(G)$ $2R(G)$;

(ii) $K(G)$ $K(G)$ (G). 4+5=9

UNIT—V

9. *(a)* Prove that every connected graph G has a spanning tree. 4

(b) Let A be the incident matrix of a tree on t vertices. Consider the t rows of A as vectors over $GF[2]$ by interpreting 0 and 1 as the elements of the two-element field $GF[2]$. Show that any t 1 rows of A are linearly independent over $GF[2]$. 5

(c) If (G) is the maximum degree of a graph G, prove that (G) 1. 5

10. *(a)* Let x be a vertex of a graph G whose degree is less than n. Prove that G is n-colourable if and only if $G \setminus x$ is n-colourable. 5

(b) If p, q are integers greater than 2, prove that

$$R(p, q) \quad R(p \quad 1, q) \quad R(p, q \quad 1)$$ 5

(c) If a graph G contains exactly one cycle of odd length, prove that (G) 3. 4

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