2016/ODD/08/23/MTM–303/477

PG Odd Semester (CBCS) Exam., December—2016

MATHEMATICS

(3rd Semester)

Course No. : MTMCC–303

(Complex Analysis)

Full Marks : 70
Pass Marks : 28

Time : 3 hours

The figures in the margin indicate full marks for the questions

Answer **five** questions, taking **one** from each Unit

UNIT—I

1. *(a)* Let $c \quad \mathbb{C}$. Prove that the function $f(z) \quad ce^{z}$ is the unique power series satisfying $f\ (z) \quad f(z)$ and $f(0) \quad c, \quad z \quad \mathbb{C}$. Further prove that

$$e^{z} \qquad e^{z}e\ , \quad z, \qquad \mathbb{C}$$ 4+3=7

(b) Verify that the function

$$f(z) \quad \begin{matrix} \mathrm{Im}\,(z^2)\,/\,|z|^2 & , & z \quad 0 \\ 0 & , \text{ if} & z \quad 0 \end{matrix}$$

satisfies the Cauchy-Riemann equations at $z \quad 0$ but f is not differentiable at $z \quad 0$. 4

(c) State whether the following is true or false with a brief justification : 3

$f(z) \quad |z \quad a|^2$ is differentiable at $z \quad a$ but it is not holomorphic at $z \quad a$.

2. *(a)* Let D be a domain in $\mathbb{C}$ with $0 \quad D$. Suppose there is a branch of the logarithm f on D. Prove that a function g defined on D is a branch of the logarithm on D if and only if $n \quad \mathbb{Z}$ such that $g(z) \quad f(z) \quad 2$ *in*, $z \quad D$. Further show that if g is a branch of the logarithm on D, then g is holomorphic on D with

$$g\ (z) \quad \frac{1}{g(z)} \qquad z \quad D$$ 5+4=9

(b) Find the point on the Riemann sphere $\mathbb{S}^2$ corresponding to the complex number

$$\frac{1}{\sqrt{2}} \quad \sqrt{2}\,i$$ 2

(c) State whether the following is true or false with a brief justification : 3

$\sin|z|$ is a holomorphic function.

UNIT—II

3. *(a)* Suppose f is an entire function and $f(\mathbb{C})$ is not dense in $\mathbb{C}$. Prove that f is a constant function. 4

(b) Let ${}_0$ and ${}_1$ be two paths in a domain D with the same initial and final points. Let be a closed form on D. If ${}_0$ is homotopic to ${}_1$ with fixed end points, prove that 7

$${}_0$$

(c) State whether the following is true or false with a brief justification : 3

There is a branch of logarithm on $\mathbb{C} \setminus \{0\}$.

4. *(a)* If is a closed path in $\mathbb{C}$, prove that the function $: \mathbb{C} \setminus \text{range}(\ ) \quad \mathbb{C}$ defined by

$$(z) \quad I(\ ,\ z),\ z \quad \mathbb{C} \setminus \text{range}(\ )$$

is a locally constant function. 7

(b) Prove that if f is an entire function such that for some $M \quad 0$ and some $k \quad \mathbb{N}$, $|f(z)| \quad M|z|^k$ for $|z|$ sufficiently large, then f is a polynomial of degree at most k. 4

(c) State whether the following is true or false with a brief justification : 3

A closed form is always exact.

UNIT—III

5. *(a)* Let f be a holomorphic function on a domain D. Prove that f is not identically equal to zero on D if and only if any compact subset of D contains only a finite number of zeros of f. 4

(b) Prove that the ring of holomorphic functions on a domain D is an integral domain with the usual addition and multiplication of functions. 5

(c) Prove that if f is a meromorphic function on $\mathbb{C}$, then f is a rational function. 5

6. *(a)* Let $z \quad a$ be an isolated singularity of a function f. Prove that the following are equivalent : 6

(i) $z \quad a$ is a removable singularity of f.

(ii) f is bounded in a deleted neighbourhood of $z \quad a$.

(iii) $\lim_{z \quad a} f(z)$ exists.

(iv) $\lim_{z \quad a} (z \quad a) f(z) \quad 0$.

(b) Let f be a continuous function on an open set $U \quad \mathbb{C}$. If f has the mean-value property and if $|f|$ has a relative maximum at a point $a \quad U$ (i.e., if $|f(z)| \quad |f(a)|$ for any z sufficiently near to a), then prove that f is constant in a neighbourhood of a. 4

(c) If P is a polynomial of degree n and f has a pole of order m at $z \quad a$, prove that the composition $P \circ f$ has a pole of order mn at $z \quad a$. 4

UNIT—IV

7. *(a)* *(i)* If f has a simple pole at $z \quad a$, show that

$$\text{Res}\,(f,\, a) \quad \lim_{z \quad a} (z \quad a) f(z)$$

(ii) If f has a pole of order k at $z \quad a$, prove that

$$\text{Res}\,(f,\, a) \quad \frac{1}{\underline{|k \quad 1}} \lim_{z \quad a} \frac{d^{k \quad 1}}{dz^{k \quad 1}} [(z \quad a)^k f(z)]$$

(iii) Evaluate

$$\frac{dz}{(z^4 \quad z^2)(z^2 \quad z \quad 2)}$$

where $\quad (t) \quad 3e^{it}, \quad 0 \quad t \quad z \quad .$ 2+3+4=9

(b) Show that all the roots of $z^5 \quad z \quad 1 \quad 0$ satisfy $|z| \quad \frac{5}{4}$. 5

8. *(a)* Prove the fundamental theorem of algebra using Rouche's theorem. 5

(b) Let D be a domain in $\mathbb{C} \setminus \{0\}$ such that there is a closed path $\quad$ in D with $I(\;,0) \quad 1$. Let $n \quad 2$, $n \quad \mathbb{N}$. Prove that there is no holomorphic function f on D such that $f^{(n)}(z) \quad z \quad\quad z \quad D$. 4

(c) Evaluate (any *one*) : 5

(i) $\displaystyle \frac{x \quad 1}{x^4 \quad 1} dx$

(ii) $\displaystyle \frac{\cos x}{1 \quad x^2} dx$

UNIT—V

9. *(a)* Find the Möbius transformation T that maps $1, i, \quad 1$ to $\quad 1, 1, i$ respectively. 4

(b) If

$$T(z) \quad \frac{az \quad b}{cz \quad d}$$

where $\quad a, b, c, d \quad \mathbb{C}$ with $\quad ad \quad bc \quad 1$, prove that $T(\mathbb{R} \quad) \quad \mathbb{R}$ if and only if we can choose a, b, c, d to be real numbers. 5

(c) Prove that an entire function f belongs to Aut ($\mathbb{C}$) if and only if there are $a, b \in \mathbb{C}$, $a \neq 0$ with $f(z) = az + b$ $\forall z \in \mathbb{C}$. 5

10. (a) Let T be a Möbius transformation with fixed points z_1 and z_2. If S is a Möbius transformation, show that $S^{-1}TS$ has fixed points $S^{-1}z_1$ and $S^{-1}z_2$. 3

(b) Show that a Möbius transformation T has 0 and ∞ as its only fixed points if and only if it is a dilation. 5

(c) If

$$T(z) = \frac{az + b}{cz + d}$$

with $ad - bc = 1$ is a Möbius transformation different from the identity map, then prove that T has either one or two fixed point(s) in $\mathbb{C}_\infty$. Further prove that T has exactly one fixed point in $\mathbb{C}_\infty$ if and only if $tr^2 A = 4$, where

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$ 6

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