PG Odd Semester (CBCS) Exam., December—2016

MATHEMATICS

(1st Semester)

Course No. : MTM–102 (C)

(Real Analysis)

Full Marks : 75
Pass Marks : 30

Time : 3 hours

The figures in the margin indicate full marks for the questions

Answer **five** questions, taking **one** from each Unit

UNIT—I

1. *(a)* Let X be a non-empty set. Prove that the following are equivalent : 6

(i) X is countable

(ii) There is a surjection $f : \mathbb{N} \quad X$

(iii) There is an injection $g : X \quad \mathbb{N}$

(b) Show that a set S is finite if and only if each non-empty subset of $\mathscr{P}(S)$ partially ordered by set inclusion has a minimal element. 5

(c) Let $A \quad x \quad \frac{1}{x} : x \quad 0$. Show that $\sup(A) \quad$ and $\inf(A) \quad 2$. 4

2. *(a)* Let X be a set and $A \quad X$. The characteristic function of the set A is defined by

$$_A(x) : \begin{cases} 1, & \text{if } x \quad A \\ 0, & \text{otherwise} \end{cases}$$

Consider the set 2^X of all functions from X to $\{0, 1\}$. Define the function

$$: \mathscr{P}(X) \quad 2^X \quad \text{by} \quad (A) : \quad _A$$

Show that is a one-to-one correspondence. 5

(b) Let X and Y be sets and let $f : X \quad Y$ be a function. Let $A \quad X$ and $B \quad Y$. Prove that—

(i) $f(f^{\ 1}(B)) \quad B$ and that equality holds if f is onto. Show by an example that the inclusion is proper in general;

(ii) $A \quad f^{\ 1}(f(A))$ and that the equality holds if f is one-to-one. Show by an example that the inclusion is proper in general. 5

(c) If $A, B \subseteq [0, \infty)$, prove that

$$\sup(AB) \leq \sup(A)\sup(B)$$

Further if $\inf(A) > 0$, prove that

$$\sup(1/A) \leq 1/\inf(A)$$

(Notation : $AB = \{ab : a \in A, b \in B\}$) 5

UNIT—II

3. (a) Show that if x is a limit point of $S \subseteq \mathbb{R}$, then for each $\varepsilon > 0$, the intersection $(x - \varepsilon, x + \varepsilon) \cap S$ is infinite. 5

(b) Show that every convergent sequence in $\mathbb{R}$ is bounded. Show with an example that a bounded sequence in $\mathbb{R}$ need not be convergent. 5

(c) Prove that every bounded sequence of real numbers has a convergent subsequence. 5

4. (a) Given any real sequence $\langle x_n \rangle$, prove that $\liminf \langle x_n \rangle \leq \limsup \langle x_n \rangle$ and $\liminf \langle x_n \rangle = \limsup \langle x_n \rangle$ if and only if $\lim \langle x_n \rangle$ exists. 5

(b) Prove that the p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent for $p > 1$ and divergent for $p \leq 1$. 5

(c) Prove that x is a limit point of a subset S of $\mathbb{R}$ if and only if there is a sequence $\langle s_n \rangle_{n \geq 1}$ in S which is not ultimately constant such that $\lim s_n = x$. 5

UNIT—III

5. (a) Show that the function

$$f(x): \begin{cases} \frac{1}{q} & \text{if } x = \frac{p}{q} \in (0, 1);\ p, q \in \mathbb{N} \text{ with } \gcd(p, q) = 1 \\ 0 & \text{if } x \in \mathbb{Q}^c \cap (0, 1) \end{cases}$$

is continuous at each $x \in \mathbb{Q}^c \cap (0, 1)$ and discontinuous at each $x \in \mathbb{Q} \cap (0, 1)$. 5

(b) Show that the function $f(x) = \sqrt{x}$ is uniformly continuous on $[0, \infty)$. 5

(c) Let $f : I \to \mathbb{R}$ be a continuous function, where I is an interval in $\mathbb{R}$. If $K \subseteq I$ is compact, prove that $f(K)$ is also compact. 5

6. (a) Let $f : I \to \mathbb{R}$ be a continuous function, where I is an interval in $\mathbb{R}$. Given $a, b \in I$ with $a < b$ and any number y_0 between $f(a)$ and $f(b)$, prove that there is a number $x_0 \in [a, b]$ such that $y_0 = f(x_0)$.

Further show that for the function

$$f(x): \quad \begin{array}{ll} \sin\frac{1}{x}, & \text{if } x \quad 0 \\ 0\ , & \text{if } x \quad 0 \end{array}$$

if $a, b \quad \mathbb{R}$ with $a \quad b$ and y_0 is a real number between $f(a)$ and $f(b)$, then there is $x_0 \quad [a, b]$ with $y_0 \quad f(x_0)$ but f is not continuous. 5+5=10

(b) Show that the polynomial

$$p(x) \quad 1 \quad x \quad \frac{x^2}{\lfloor 2} \quad \ldots \quad \frac{x^n}{\lfloor n}$$

has a unique real root if n is odd and no real roots if n is even. 5

UNIT—IV

7. *(a)* Let $f : \mathbb{R} \quad \mathbb{R}$ be uniformly continuous function and define

$$f_n(x): \quad f \quad x \quad \frac{1}{n} \qquad n \quad \mathbb{N}$$

and $x \quad \mathbb{R}$. Show that the sequence of functions $\langle f_n \rangle$ converges uniformly to f on $\mathbb{R}$. 5

(b) If $\sum_{n \quad 1} a_n x^n$ has radius of convergence $r \quad 1$, find the radii of convergence of the series $\sum_{n \quad 1} (a_n)^2 x^n$ and $\sum_{n \quad 1} a_n x^{2n}$. 4

(c) Prove that the sequence of functions $\langle f_n \rangle$, where $f_n(x) \quad x^n, \quad x \quad [0, 1]$, converges pointwise on [0, 1] to the limit function

$$f(x) \quad \begin{array}{ll} 0, & 0 \quad x \quad 1 \\ 1, & x \quad 1 \end{array}$$

but the convergence is not uniform. 6

8. *(a)* Show by an example with a brief justification that a uniform limit of a sequence of differentiable functions may not by differentiable. 5

(b) Show that $\langle (\sin nx) / (1 \quad nx) \rangle$ converges uniformly on $[a, \quad)$ for any $a \quad 0$ but not for $a \quad 0$. 5

(c) Show that the power series

$$\sum_{n \quad 1} a_n x^n$$

where

$$a_n \quad \begin{array}{ll} 1 & \text{if } n \text{ is odd} \\ n & \text{if } n \text{ is even} \end{array}$$

has radius of convergence 1, and that $\lim_n \frac{a_{n \quad 1}}{a_n}$ does not exist. 5

UNIT—V

9. *(a)* *(i)* Give examples with a brief justification that the sum and product of two monotonic functions need not be monotonic.

(ii) If f and g are functions of bounded variations on $[a, b]$, prove that f g and fg are of bounded variations.

3+6=9

(b) Prove that a monotonic function $f : I$ $\mathbb{R}$, where I is an interval in $\mathbb{R}$, has only jump discontinuities (i.e., discontinuities of the first kind). Further show that the set D $\{x$ $I : f$ is discontinuous at $x\}$ is countable. 6

10. *(a)* If f is a function continuous on $[a, b]$ with a bounded derivative on (a, b), prove that f is of bounded variation. 5

(b) Prove that every monotonic function on $[a, b]$ is of bounded variation. 5

(c) Show that the function

$$f(x): \quad \begin{array}{ll} x\sin\left(\frac{}{2x}\right), & \text{if } x \quad (0, 1] \\ 0 \quad , & \text{if } x \quad 0 \end{array}$$

is continuous on [0, 1] but not of bounded variation. 5

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