PG Odd Semester (CBCS) Exam., December—2016

MATHEMATICS

(3rd Semester)

Course No. : MTM–301 (C)

(Real Analysis—II)

Full Marks : 75
Pass Marks : 30

Time : 3 hours

The figures in the margin indicate full marks for the questions

Answer **five** questions, selecting **one** from each Unit

UNIT—I

1. *(a)* Let be a non-empty open subset of $\mathbb{R}^n$ and let f : $\mathbb{R}$. Show that if f is differentiable at X_0 , then for each v(0) $\mathbb{R}^n$, the directional derivative $f_v(X_0)$ exists and

$$f_v(X_0) \quad f(X_0)(v)$$ 5

(b) Check the differentiability of the function $f : \mathbb{R}^2 \quad \mathbb{R}^3$ defined by

$$f(x, y) \quad (x^2y, x^2 \quad y^3) \text{ at } (0, 0)$$ 5

(c) For the function $f : \mathbb{R}^2 \quad \mathbb{R}$ defined by

$$f(x, y) \quad \begin{cases} \dfrac{xy(x^2 \quad y^2)}{x^2 \quad y^2} & \text{if } (x, y) \quad (0, 0) \\ 0 & \text{if } (x, y) \quad (0, 0) \end{cases}$$

show that $f_{xy}(0, 0) \quad f_{yx}(0, 0)$. 5

2. *(a)* Show that the function $f : \mathbb{R}^2 \quad \mathbb{R}$ defined by

$$f(x, y) \quad \begin{cases} \dfrac{xy^2}{x^2 \quad y^4} & \text{if } (x, y) \quad (0, 0) \\ 0 & \text{if } (x, y) \quad (0, 0) \end{cases}$$

is not differentiable at (0, 0) although $f_v(0, 0)$ exists for each

$$v \quad (a, b)(\quad (0, 0)) \quad \mathbb{R}^2$$ 5

(b) Prove or disprove : If $f : \mathbb{R}^n \quad \mathbb{R}$ is differentiable at $X_0 \quad \mathbb{R}^n$, then all the partial derivatives of f are continuous at X_0. 5

(c) Show that the function $f : \mathbb{R}^2 \quad \mathbb{R}$ defined by

$$f(x, y) \quad \begin{cases} xy\dfrac{x^2 \quad y^2}{x^2 \quad y^2} & \text{if } (x, y) \quad (0, 0) \\ 0 & \text{if } (x, y) \quad (0, 0) \end{cases}$$

is differentiable at (0, 0). 5

UNIT—II

3. *(a)* Show that the function $f : \mathbb{R}^3 \quad \mathbb{R}$ defined by

$f(x, y, z) \quad (x \quad y \quad z)^3 \quad 3(x \quad y \quad z) \quad 24xyz$
for all $(x, y, z) \quad \mathbb{R}^3$

has a local minimum at (1, 1, 1) and a local maximum at (1, 1, 1). 5

(b) Find the shortest distance from the origin to the hyperbola

$x^2 \quad 8xy \quad 7y^2 \quad 225, \quad z \quad 0$ 5

(c) Let $f : \mathbb{R}^2 \quad \mathbb{R}$ be continuously differentiable and also let $f(x_0, y_0) \quad 0$, and suppose that $\frac{f}{y}(x_0, y_0) \quad 0$. Show that there is an open interval $(y_0 \quad , y_0 \quad)$ and a continuous function

$g : (y_0 \quad , y_0 \quad) \quad \mathbb{R}$

such that $g(x_0) \quad y_0$ and $f(x, \ g(x)) \quad 0$. 5

4. *(a)* Determine the critical points and the local maxima and minima of the function $f(x, y) \quad x^3 \quad 3axy \quad y^3$ for all values of a. 5

(b) State and prove inverse function theorem for a function $f : \mathbb{R}^n \quad \mathbb{R}^n$. 1+9=10

UNIT—III

5. *(a)* Show that every continuous function $f : [0, 1] \quad \mathbb{R}$ is Riemann integrable. 5

(b) Let A be a countably infinite subset of $\mathbb{R}$. Show that A is Lebesgue measurable. 5

(c) Show that there exists a subset $N \quad (0, 1)$ such that, N is not a Lebesgue measurable set. 5

6. *(a)* Show that the function $f : [0, 1] \quad \mathbb{R}$ defined by

$$f(x) \quad \begin{array}{l} x \text{ if } x \quad \mathbb{Q} \quad [0, 1] \\ 0 \text{ if } x \quad (\mathbb{R} \setminus \mathbb{Q}) \quad [0, 1] \end{array}$$

is not Riemann integrable. 5

(b) Evaluate the limit

$$\lim_n \quad 1 \quad \frac{1}{n} \quad 1 \quad \frac{2}{n} \quad 1 \quad \frac{3}{n} \quad \cdots \quad 1 \quad \frac{n}{n} \quad ^{1/n}$$

with the help of Riemann's definition of integration. 5

(c) Let G be a non-empty open subset of $\mathbb{R}$. Show that $m(G) \quad 0$. Is the above true for any non-empty closed subset of $\mathbb{R}$? Justify. 3+2=5

UNIT—IV

7. *(a)* Show that a function $f : \mathbb{R} \to \mathbb{R}$ is Lebesgue measurable iff for every Borel subset B of $\mathbb{R}$, $f^{-1}(B)$ is a Lebesgue measurable set. 5

(b) Show that

$\mathcal{A} = \{A \subseteq \mathbb{R} : A \text{ is countable or } \mathbb{R} \setminus A \text{ is countable}\}$

is a σ-algebra of subsets of $\mathbb{R}$. 5

(c) Show that every Riemann integrable function $f : [a, b] \to \mathbb{R}$ is Lebesgue integrable. 5

8. *(a)* Show that the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} \frac{1}{x} & \text{if } x \in \mathbb{R} \setminus \{0\} \\ 0 & \text{if } x = 0 \end{cases}$$

is Lebesgue measurable. 3

(b) Show that the Borel σ-algebra $\mathcal{B}(\mathbb{R})$ is generated by the class

$\{[a, b) : a, b \in \mathbb{Q}, a < b\}$ 5

(c) For each $x \in [0, 1]$, let

$$f(x) = \begin{cases} 1 - x & \text{if } x = \frac{1}{5^n} \text{ for some } n \in \mathbb{N} \\ x^2 & \text{if } x = \frac{1}{7^n} \text{ for some } n \in \mathbb{N} \\ 5x^3 & \text{if } x = \frac{1}{11^n} \text{ for some } n \in \mathbb{N} \\ x^4 & \text{if otherwise} \end{cases}$$

evaluate the Lebesgue integral $\int_{[0,1]} f \, dm$. 4

(d) Prove or disprove : 3

$f : [0, 1] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \cap [0, 1] \\ 0 & \text{if } x \in (\mathbb{R} \setminus \mathbb{Q}) \cap [0, 1] \end{cases}$$

is Lebesgue integrable.

UNIT—V

9. *(a)* State and prove Lebesgue's monotone convergence theorem. 1+6=7

(b) Using monotone convergence theorem, evaluate

$$\int_{[1, \infty)} \frac{1}{x^2} dx$$ 4

(c) Using dominated convergence theorem, evaluate

$$\lim_{n} \int_0^1 \frac{n\sqrt{x}}{1 + n^2x^2}\,dx$$ 4

10. *(a)* State and prove Lebesgue's dominated convergence theorem. 1+6=7

(b) State and prove Fatou's lemma. 1+4=5

(c) Evaluate the Lebesgue integral

$$\int_{(0,\ )} f\,dm$$

where

$$f(x) \quad \begin{cases} \dfrac{1}{\sqrt{x}} & \text{if } 0 \quad x \quad 1 \\ \dfrac{1}{x} & \text{if } x \quad 1 \end{cases}$$ 3

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