PG Odd Semester (CBCS) Exam., December—2016

MATHEMATICS

(3rd Semester)

Course No. : MTM–303 (C)

(Complex Analysis)

Full Marks : 75
Pass Marks : 30

Time : 3 hours

The figures in the margin indicate full marks for the questions

Answer **five** questions, selecting **one** from each Unit

UNIT—I

1. *(a)* Let $f : D \quad \mathbb{C}$ be a holomorphic function, where D is a domain in $\mathbb{C}$. Prove that f is constant on D if and only if $f \quad 0$ on D. Further show with the help of an example that if D is not connected, the above result may fail. 5

(b) Show that Cauchy-Riemann equation are necessary but not sufficient conditions for a function f of a complex variable to be holomorphic on an open set U of $\mathbb{C}$. (A brief justification is required.) 5

(c) Prove that—

(i) $e^z \quad e^z e \quad z, \quad \mathbb{C}$;

(ii) $|e^z| \quad e^{\mathrm{Re}(z)}, \quad z \quad \mathbb{C}$. 4+1=5

2. *(a)* Let

$$f(z) \quad \underset{n \quad 0}{a_n z^n}$$

have radius of convergence 0. Prove that—

(i) $k \quad \mathbb{N}$, the power series

$$h_k(z) \quad \underset{n \quad k}{n(n \quad 1)\cdots(n \quad k \quad 1)a_k z^{n \quad k}}$$

has radius of convergence ;

(ii) the function f is infinitely differentiable on the disc

$$D(0, \ ) : \ \{z \quad \mathbb{C} : |z| \quad \}$$

and $f^{(k)}(z) \quad h_k(z), \quad k \quad \mathbb{N}$ and $z \quad D(0, \ )$;

(iii) $n \quad \mathbb{N} \quad \{0\}, \ a_n f^{(n)}(0) / \underline{\lfloor n} \ .$

4+5+1=10

(b) If $f(z)$ is a branch of the logarithm on a domain $D \quad \mathbb{C} \setminus \{0\}$, prove that f is holomorphic on D with

$$f'(z) \quad \frac{1}{z}, \quad z \quad D$$

5

UNIT—II

3. *(a)* Let be a differential one-form on a domain D in $\mathbb{C}$. Prove that for any p.d.p. in U, the integral depends only at the end points of if and only if 0, closed p.d.p. in U. 4

(b) If : $[a, b]$ U is a path in an open set U $\mathbb{C}$ and Pdx Qdy is a closed differential one-form on U, where P, Q are functions of x and y, prove that there is a primitive of along , which is unique up to the addition of a constant. 6

(c) If is a closed path in $\mathbb{C} \setminus \{0\}$, prove that

$$\frac{1}{2\ i}\quad\frac{dz}{z}\quad\mathbb{Z}$$

5

4. *(a)* Let $_0$ and $_1$ be two paths in a domain D and let be a closed form on D. If $_0$ is homotopic to $_1$ with fixed end points, prove that

$$_0 \qquad _1$$

6

(b) If $_0$, $_1$: $[a, b]$ $\mathbb{C}$ are closed paths with

$$0 \quad |\ _1(t)| \ |\ _0(t)| \quad t \quad [a, b]$$

prove that $I(\ _1 \quad _0, 0)$ $I(\ _0, 0)$. 4

(c) If is a closed path in $\mathbb{C}$, then prove that the function : $\mathbb{C} \setminus$ range( ) $\mathbb{Z}$ defined by ( ) $I(\ ,\ )$ is a locally constant function. 5

UNIT—III

5. *(a)* Let f be a holomorphic function on a domain D. Prove that f is not identically zero on D if and only if any compact subset of D contains at most a finite number of zeros of f. 5

(b) If f is a non-constant holomorphic function on a domain D, prove that $|f|$ has no maximum in D. 5

(c) Let z a be an isolated singularity of a function f. Prove that the following are equivalent : 5

(i) z a is a removable singularity of f

(ii) f is bounded in a deleted neighbourhood of f

(iii) $\lim_{z \ a} f(z)$ exists

(iv) $\lim_{z \ a} (z \quad a) f(z) \quad 0$

6. *(a)* Let D be a domain in $\mathbb{C}$ with $a \in D$. If f is a holomorphic function on $D \setminus \{a\}$ with a pole at $z = a$, prove that $m \in \mathbb{N}$ such that $(z - a)^m f(z)$ has a removable singularity at $z = a$. 5

(b) Let $z = a$ be an isolated singularity of a function f and let

$$f(z) = \sum_{n} a_n (z - a)^n$$

be its Laurent series expansion in the annulus $0 < |z - a| < R$. Prove that—

(i) $z = a$ is a removable singularity of f if and only if $a_n = 0 \ \forall \ n \geq 1$;

(ii) $z = a$ is a pole of order m of f if and only if $a_m \neq 0$ and $a_n = 0$, $n > m \geq 1$. 5+5=10

UNIT—IV

7. *(a)* Prove the fundamental theorem of Algebra using Rouch's theorem. 5

(b) Evaluate

$$\int_{\gamma} \frac{dz}{z^2(z^2 - z - 2)}$$

where $\gamma(t) = 3e^{it}$, $0 \leq t \leq 2\pi$. 5

(c) Determine the number of solutions of $e^z = 2z - 1$ with $|z| < 1$. 5

8. *(a)* Evaluate any *one* of the following : 5

(i) $$\int_{-\infty}^{\infty} \frac{x - 1}{x^4 - 1} dx$$

(ii) $$\int_{-\infty}^{\infty} \frac{\cos x}{1 - x^2} dx$$

(b) Prove the open mapping theorem with the help of Rouch's theorem. 5

(c) Show that all the roots of $z^5 - z - 1 = 0$ satisfy $|z| < \frac{5}{4}$. 5

UNIT—V

9. *(a)* Prove that a Möbius transformation

$$T(z) = \frac{az + b}{cz + d}, \ ad - bc \neq 0$$

has at most two distinct fixed points in $\mathbb{C}$ unless $T(z) \equiv z$ identically. 4

(b) Prove that an entire function f belongs to Aut $(\mathbb{C})$ if and only if there is $a, b \in \mathbb{C}$, $a \neq 0$ with $f(z) = az + b$, $\forall \ z \in \mathbb{C}$. 6

(c) Show that a Möbius transformation has 0 and as its only fixed points if and only if it is a dilation. 5

10. *(a)* Find the Möbius transformation that maps 1, *i*, *i* to 1, *i*, 1 respectively. 5

(b) If

$$T(z) \quad \frac{az \quad b}{cz \quad d}$$

is a Möbius transformation, where $a, b, c, d \quad \mathbb{C}$, $ad \quad bc \quad 1$, prove that $T(\mathbb{R}) \quad \mathbb{R}$ if and only if we can choose a, b, c, d to be real numbers. 5

(c) Prove that a Möbius transformation T has as its only fixed point if and only if it is a translation different from the identity. 5

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