PG (CBCS) ODD SEMESTER EXAMINATION, 2022

MATHEMATICS

3rd Semester

Course No. : MTMCC - 305

(Complex Analysis)

Full Marks : 70

Pass Marks : 28

Time : 3 hours

The figures in the margin indicate full marks for the questions

(Answer any five questions, taking one from each unit)

UNIT - I

1. (a) Let z, w be non zero complex numbers. Show that arg (zw) = arg (z) + arg (w). Also, prove or disprove that 2+2=4
Arg (zw) = Arg (z) + Arg (w)

(b) Let $f : E \subset \mathbb{C} \to \mathbb{C}$ be $\mathbb{C}$-differentiable at $z_0 = x_0 + iy_0 \in E^0$. Show that f is $\mathbb{R}$-differentiable at (x_0, y_0) and that
$u_x (x_0, y_0) = v_y (x_0, y_0)$ and $u_y (x_0, y_0) = -v_x (x_0, y_0)$
where u = Ref and v = Imf. 5

(c) Let G be an open subset of $\mathbb{C}$ and let $f : G \to \mathbb{C}$ be differentiable at $z_0 \in G$. Show that the function $g : G \to \mathbb{C}$ defined by

$g(z) = f(z)$ for all $z \in G$ is differentiable at z_0 iff $f'(z_0)=0$ 5

2. (a) Check if the function $f : \mathbb{C} \to \mathbb{C}$ defined by

$$f(z) = \begin{cases} \frac{z^5}{|z|^4}, & If\ z \neq 0 \\ 0, & If\ z = 0 \end{cases}$$

is $\mathbb{C}$ -differentiable at 0. 5

(b) Let Ω be a domain in $\mathbb{C}$ and let $f : \Omega \to \mathbb{C}$ be a holomorphic function. If L is a straight line in $\mathbb{C}$ and $f(\Omega) \subset$ L, show that f is a constant function. 4

(c) Show that the function given by

$$u(x,y) = \frac{1}{2} \log (x^2 + y^2)\ \forall\ (x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$$

does not have a harmonic conjugate. 5

UNIT - II

3. (a) Let R be the radius of convergence of a power series $\sum_{n=0}^{\infty} a_n z^n$, where $a_n \neq 0\ \ \forall\, n \in \mathbb{N} \cup \{0\}$ show that $R = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|$, provided the limit exists. 4

(b) Show that if $f : \Omega \to \mathbb{C}$ is a branch of logarithm on a domain Ω of $\mathbb{C}$, then f is holomorphic and

$$f'(z) = \frac{1}{z}\ \forall\ z \in \Omega$$ 5

(c) State and prove Liouville's theorem. 1+4=5

(i) Show that there exists a sequence (K_n) of compact subsets of G such that $G = \bigcup_{n\in\mathbb{N}} K_n$ and $K_n \subseteq K^0_{n+1}$ for all $n \in \mathbb{N}$.

(ii) If K is a subset of G, then there exists $n \in \mathbb{N}$ such that $K \subset K_n$. Show it.

(iii) Prove that the function $\rho : \subset (G, X) \times \subset (G, X) \to \mathbb{R}$ defined by

$$\rho(f,g) = \sum_{n=1}^{\infty} \frac{1}{2^n} \cdot \frac{\rho_n(f,g)}{1+\rho_n(f,g)}\ \forall\, f,g\ \in C(G,X)$$

where $\forall\ n \in \mathbb{N}$, $\rho_n(f, g) = \sup \{ d(f(z), g(z) : z \in K_n\}$ is a metric on C (G, X).

(b) Let G be an open subset of $\mathbb{C}$. If (f_n) is a sequence in H(G) and $f \in \subset (G, \mathbb{C})$ such that $f_n \to f$, then show that f is holomorphic and that for each $k \in \mathbb{N}$ $f_n^{(K)} \to f^{(K)}$. 5

10. (a) State and prove Hurzitz's theorem. 1+5=6

(b) Let G be an open subset of $\mathbb{C}$ and let (X, d) be a complete metric space. Show that a set $\mathfrak{F} \subset C(G, X)$ is normal iff the following conditions are satisfied. 7

(i) $\forall\, z \in G, \{f(z) : f \in \mathfrak{F}\}$ has compact closure in X.

(ii) $\mathfrak{F}$ is equicontinuous at each point of G.

(c) State Riemann mapping theorem. 1

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4. (a) Let the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n z^n$ be $R > 0$ show that the function $f : B_R(O) \to \mathbb{C}$ defined by $f(z) = \sum_{n=0}^{\infty} a_n z^n \quad \forall z \in B_R(0)$ is holomorphic. 5

(b) State uniqueness theorem. Using uniqueness theorem show that $\sin^2 z + \cos^2 z = 1 \quad \forall z \in \mathbb{C}$ 1+3=4

(c) If $u : \mathbb{R}^2 \to \mathbb{R}$ is a bounded harmonic function, show that u is constant. 5

UNIT - III

5. (a) Show that every non-constant entire function is open. 5

(b) Let z_0 be an isolated singulartrty of a function f. Show that z_0 is an essential singularity of f if and only if for each $\delta > 0$, the set

$F = \{w \in \mathbb{C} : \exists\ z \in B_\delta(z_0) - \{z_0\}$ such that $f(z) = w\}$ is dense in $\mathbb{C}$. 5

(c) State and prove Schwarz lemma. 1+3=4

6. (a) Show that every zero of a holomorphic function is isolated. 4

(b) Determine all the singularities and their nature for the function given by $f(z) = \frac{\sin^2 z}{z^3}$ for all $z \in \mathbb{C} - \{O\}$. 5

(c) Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function such that $\text{Re } f(z) > 0$ for all $z \in \mathbb{C}$. Show that f is a constant function. 5

UNIT - IV

7. (a) Using Rouche's theorem show that every non-constant polynomial of degree n in $\mathbb{C}$ has exactly n number of zeros counted with multiplicities. 5

(b) Using calculus of resideu show that 5

$$\int_0^{\infty} \frac{\sin x}{x}\, dx = \pi/2$$

(c) Show that the number of roots of the equation $z^4 + 3z^3 + 6 = 0$ inside the circle $|z| = 2$ is 3. 4

8. (a) State and prove argument principle. 1+4=5

(b) Using calculus of residue, evaluate 5

$$\int_{-\infty}^{\infty} \frac{x \sin \pi x}{x^2 + 2x + 5}\, dx$$

(c) Find Laurent expansion of the function given by $f(z) = \frac{1}{z(z^2+1)}$ valied for $|z-3| > 2$. 4

UNIT - V

9. (a) Let G be an open subset of $\mathbb{C}$ and let (X, d) be a complete metric space. 3+3+3=9