8. (a) State and prove the Eisenstein's Criterion for irreducibility over Q. 5

   (b) Lt K/F be a field extension and $a \in k$ be algebraic over F. Show that there exists a unique monic irreducible polynomial $p(x) \in F[X]$ such that $p(a) = 0$. 5

   (c) Find the minimal polynomial of $\sqrt{2} + \sqrt{3}$ over Q. 4

UNIT - V

9. (a) State and prove Kroneker's theorem. 6

   (b) Show that there exists a splitting field for any non-constant polynomial over any field. 4

   (c) Let F be a finite field. Show that char F = p and F is an extension of $\mathbb{Z}_p$, where p is a prime number. 4

10. (a) Show that for every prime p and for any positive integer $n \geq 1$, there exists a unique field with p^n elements. 5

    (b) Show that any two finite fields of the same order are isomorphic. 5

    (c) Find a primitive element of the extension $Q(\sqrt{2}, \sqrt{3}) | Q$. 4

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PG ODD SEMESTER EXAMINATION, 2022

MATHEMATICS

1st Semester

Course No. : MATCC - 501

(Basic Abstract Algebra)

Full Marks : 70

Pass Marks : 28

Time : 3 hours

The figures in the margin indicate full marks for the questions

(Answer any five questions, taking one from each unit)

UNIT - I

1. (a) Show that disjoint cycles commute with one another. 4

   (b) Let G be a cyclic group of order n>1 and let d be any positive divisor of n. Show that G has a unique sub-group of order d. 6

   (c) Can a group of order 55 have exactly 20 elements of order 11? Justify your answer. 4

2. (a) Establish Lagrange's theorem for finite groups. 5

(b) Show that $x^2 = (1\ 2\ 3\ 4)$ has no solution in S_7 but $x^3 = (1\ 2\ 3\ 4)$ has atleast two solutions in S_7. 4

(c) Examine if the groups (Q, +) and (Q*, .) are cyclic. 2+3=5

UNIT - II

3. (a) Let N_1 and N_2 be normal subgroups of the groups G_1 and G_2 respectively. Show that

(i) $N_1 \times N_2$ is a normal subgroup of $G_1 \times G_2$.

(ii) $G_1 \times G_2 / N_1 \times N_2 \cong G_1/N_1 \times G_2/N_2$ 3+3=6

(b) Let G be a group. Show that a non-empty set X is a G-set if and only if there is a homomorphism from G to S(X). 4

(c) Let p be a prime number and let G be a group of order p^n. Show that if a finite set X is a G-set, then $|X| \equiv |X_G| \pmod p$, whre $X_G = \{ x \in X : g\,x = x\ \forall\ g \in G \}$. 4

4. (a) Let G be an internal direct product of its normal subgroups H_1 and H_2 show that 4

$$G / H_1 \cong H_2$$

(b) State and prove Sylow's third theorem. 1+4=5

(c) Find the number of distinguishable ways the edges of an equilateral triangle can be painted if four different colours of paint are available, assuming only one colour is used on a single edge and the same colour may be used on different edges. 5

UNIT - III

5. (a) Let R be a comutative ring with unity. Show that the ideal M of R is maximal ideal if and only if R/M is a field. 6

(b) Write each of ideals <6> + <8>, <6> ∩ <8> and <6> <8> of the ring of integers as a principal ideal. Also, find the smallest ideal containing <6> ∪ <8>. 4

(c) Prove or disprove: A principal ideal domain is a field. 4

6. (a) Let p be a prime number and I be an ideal generated by p and x, i.e. $\langle p, x \rangle$ in $\mathbb{Z}[X]$. Show that I is Maximal Ideal of Z[X]. 5

(b) Show that Z[X] is not a Principal Ideal Domain (PID). 5

(c) Prove that $Z_4[X]$ has infinitely many units and nilpotent elements. 4

UNIT - IV

7. (a) Show that every non-zero, non-unit element in a PID is divisible by an irreducible element. 5

(b) Show that every irreducible element, in a Unique Factorization Domain (UFD) is a prime element. 5

(c) Test whether the polynomial $f(x) = 1 + px + p^{n-1} x^n$, where p is any prime, is irreducible over Q. 4