(b) Consider the following basis of Euclidean space R^3 :{ v_1 = (1, 1, 1) , v_2 = (0, 1, 1), v_3 = (0, 0, 1)}. Using the Gram-schmidt orthogonalization process, transform the basis $\{v_i\}$ into an orthonormal basis. 7

UNIT - V

9. (a) When of the following functions f, defined on vectors $\alpha = (x_1, x_2)$ and $\beta = (y_1, y_2)$ in R^2 are bilinear form?

(i) $f(\alpha, \beta) = x_1y_2 - x_2y_1$ 7

(ii) $f(\alpha, \beta) = (x_1 - y_1)^2 + x_2y_2$

(b) If U is an n-dimensional vector spece with basis $\{ \alpha_1, \ldots\ldots, \alpha_n \}$, if V is an m-dimensional vector space $\{ \beta_1, \ldots., \beta_m\}$, and if $\{a_{ij}\}$ is any set of n^m scatars (i = 1, ..., n ; j = 1, ..., m) then prove that there is one and only one bilinear form f on U⊕V such that $f(\alpha_i, \beta_j) = a_{ij}\ \forall\ i$ and j. 7

10. (a) If V is a finite dimensional vector space, then a bilinear form f on V is symmetric if and only if its matrix A in some (or every) ordred basis is symmetric i.e. $A = A^1$. 7

(b) Let f be a non-degenerate bilinear form on a finite dimensional vector space V. Prove that the set G of all linear operators on V which preserve f is a group under the operation of composotion. 7

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PG (CBCS) ODD SEMESTER EXAMINATION, 2022

MATHEMATICS

3rd Semester

Course No. : MTMCC - 302

(Linear Algebra)

Full Marks : 70

Pass Marks : 28

Time : 3 hours

The figures in the margin indicate full marks for the questions

(Answer any five questions, taking one from each unit)

UNIT - I

1. (a) Let U and V be finite dimensional vector spaces over the field F and let T be a linear transformation from U into V. Then prove that

$$rank\ (T) + nullity\ (T) = dim\ U$$ 7

(b) Show that the mappings $T : V_2(R) \to V_3(R)$ defined as

$$T(a, b) = (a + b,\ a - b\ ,\ b)$$

is a linear transformation from $V_2(R)$ into $V_3(R)$. Find the range, rank, null-space and nullity of T. 7

2. (a) Let V be a vector space over the field F. Then

prove that the set Hom (V, F) of all linear functionals on V is also a vector space over the field F. 7

(b) The vectors $\alpha_1 = (1, 1, 1)$, $\alpha_2 = (1, 1, -1)$ and $\alpha_3 = (1, -1, -1)$ form a basis of $V_3(\mathbb{R})$. If $\{f_1, f_2, f_3\}$ is the dual basis and if $\alpha = (0, 1, 0)$ find $f_1(\alpha)$, $f_2(\alpha)$ and $f_3(\alpha)$. 7

UNIT - II

3. (a) State and prove the primary decomposition theorem. 7

(b) Prove that a linear operator $T : V \to V$ can be represented by a diagonal matrix if and only if the vector space V has a basis consisting of eigen vectors of T. 7

4. Suppose W is invariant subspace under a linear operator $T : V \to V$. Prove that T induces a linear operator T_1 on V/W defined by $T_1(v+W) = T(v)+W$. Morever, if T is a zero of any polynomial, then show that T_1 is also a zero of the same polynomial. Deduce that minimal polynomial of T_1 divides the minimal polynomial of T. 14

UNIT - III

5. (a) Let $T : V \to V$ be a linear operator whose characteristic polynomial factors into linear polynomials. Then prove that V has a basis in which T is represented by a triangular matrix. 7

(b) Let A be a 4x4 matix with minimal polynomial $m(t) = (t^2+1)(t^2-4)$. Find the rational canonical form for A if A is a matrix over

(i) the rational field $\mathbb{Q}$,

(ii) the real field $\mathbb{R}$,

(iii) the complex field $\mathbb{C}$.

6. (a) State and prove Jordan canonical form. 7

(b) Determine all possible Jordan canonical forms for a linear operator $T:V \to V$ whose characteristics polynomial is $\Delta(T) = (t-3)^2 (t-2)^3$.

UNIT - IV

7. (a) State and prove Gram-Schmidt orthogonalization process. 7

(b) If in an inner product space $||\alpha+\beta|| = ||\alpha|| + ||\beta||$. then prove that the vectors α and β are linearly dependent. Give an example to show that the converse of this statement is false. 7

8. (a) If $\{ w_1, ..., w_n\}$ is a finite orthonormal set in V, then prove that

$$\sum_{i=1}^{m} |(w_i, v)|^2 \leq ||v||^2 \text{ for all } v \in V$$

Further more, prove that equality holds if and only if v is in the subspace spanned by w_1, ..., w_n. 7