(b) Prove or disprove : The pointwise limit of a sequence of Riemann integrable functions is Riemann integrable. 4

(c) Show that the uniform limit of a sequence of uniformly continuous functions is uniformly continuous. 5

10. (a) Let for each $n \in \mathbb{N}$
$f_n : \mathbb{R} \to \mathbb{R}$ be given by

$$f_n(x) = \begin{cases} 1 & if \quad -n \le x < n \\ 0 & if \quad |x| > n \end{cases}$$

show that (f_n) converges pointwise to $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = 1 \ \forall \ x \in \mathbb{R}$. Is this converges uniform? Justify. 3+2=5

(b) Let (f_n) be a sequence of functions where $\forall \ n \in \mathbb{N}$ $f_n(x) = nx$ for all $x \in \mathbb{R}$. Does the seque (f_n) converge pointwise? Justify. 3

(c) Prove or disprove : If (f_n) is a seque of differentiable functions. Such that (f_n) converges uniformly to f, then f is differentiable.

(d) Define non-uniform convergence of a sequence of functions (f_n), when $\forall \ n \in \mathbb{N}$ $f_n : \mathbb{R} \to \mathbb{R}$. 2

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PG EVEN SEMESTER EXAMINATION, 2023

MATHEMATICS

2nd Semester

Course No. : MAT - 551

(Basic Real Analysis)

Full Marks : 70

Pass Marks : 28

Time : 3 hours

The figures in the margin indicate full marks for the questions

(Answer one question from each unit)

Unit - I

1. (a) Let l^∞ denote the set of all bounded sequences in $\mathbb{R}$ and let $d_\infty : l^\infty \times l^\infty \to \mathbb{R}$ be defined by $d_\infty(x, y) : \sup \{|x_n - y_n| : n \in \mathbb{N}\}$ for all $x = (x_n), y = (y_n) \in l^\infty$. Show that d_∞ is a metric on l^∞. What is the induced metric on the subset of l^∞ consisting of all sequences having 0 and 1 as the only entries? 3+1=4

(b) Show that union of any two closed sets is closed in a metric space. Does the result remain true for infinite union? Justify your answer. 2+2=4

(c) Let F be a nonempty subset in a metric space (X, d). Show that $\bar{F} = \{ x \in X : d(x, F) = 0 \}$ 3

(d) Let X be a metric space and let $A \subset X$ be such that $\delta(A) < r$ where $\delta(A) = \sup\{d(x, y) : x, y \in A\}$. If the open ball $B_r(x)$ interests A, show that $A \subset B_{2r}(x)$. 3

2. (a) Show that a set in a metric space is closed iff it contains its boundary. 4

(b) If F is a closed subset in a metric space (X, d) and $x \in X - F$, then show that $d(x, F) > 0$. Show further that there exist open sets G, H such that $x \in G$, $F \subset H$ and $G \cap H = \phi$ 2+2=4

(c) Let X be a metric space. If $x, y \in X$ and r, s > 0 such that $B_r(x) \subsetneq B_s(y)$, then show that $r < 2s$ 3

(d) Prove or disprove : If F_1 , F_2 are closed subsets of a metric space (X, d) such that $F_1 \cap F_2 = \phi$, then $d(F_1, F_2) > 0$. 3

Unit - II

3. (a) Let (x_n) and (y_n) be Cauchy sequences in a metric space (X, d). Show that the sequence $(d(x_n, y_n))$ converges in $\mathbb{R}$ equipped with the usual metric. 4

(b) State and prove Cantor's intersection theorem. 1+4=5

(c) Is it possible to define a metric d on $\mathbb{R}$ such that the sequence $\left(\frac{1}{n}\right)$ does not converge in $(\mathbb{R}, d)$? Justify. 3

8. (a) Prove or disprove : If $f : \mathbb{R} \to \mathbb{R}$ is differentiable, then for each $x \in \mathbb{R}$, there exist $a, b \in \mathbb{R}$ with $a < x < b$ such that $f'(x)(b - a) = f(b) - f(a)$. 3

(b) Let $f : [0, 2] \to \mathbb{R}$ be define by

$$f(x) = \begin{cases} 1 & if \quad 0 \le x < 1 \\ 0 & if \quad 1 \le x \le 2 \end{cases}$$

show that $F : [0, 2] \to \mathbb{R}$ defined by

$$f(x) = \int_0^x f(t)\, dt \quad \forall\, x \in [0, 2]$$

is differentiable but $F'(1) \neq f(1)$. Does it violate the fundamental theorem of Calculus? Justify. 3+1=4

(c) Prove or disprove : Every Riemann integrable function. $f : [0, 1] \to \mathbb{R}$ is continuous. 4

(d) Find the exact number of real roots for the equation $x^7 + 12x + 5 = 0$

Unit - V

9. (a) Show that the sequence of functions (f_n), where $\forall\, n \in \mathbb{N}$, $f_n : (0, 1) \to \mathbb{R}$ given by $f_n(x) = x^n \;\forall\, x \in (0, 1)$, converges pointwise to the function $f : (0, 1) \to \mathbb{R}$ given by $f(x) = 0 \quad \forall\, x \in (0, 1)$. Is the convergence uniform ? Justify. 3+2=5

(d) Give an example with justification of an incomplete metric space. 2

4. (a) Let (X, d) be a metric space and $Y \subset X$. If (Y, d_Y) is complete show that Y is closed in X. Is the converse true? Justify. 2+2=4

(b) State and prove Baire's theorem for complete metric spaces. 1+5=6

(c) Show that the metrics d_1 and d_2 on $\mathbb{R}$ are equivalent but not strongly equivalent, where

$d_1(x,y) = |x-y|$ and $d_2(x,y) = \dfrac{|x-y|}{1+|x-y|}$

for all $x, y \in \mathbb{R}$. 2+2=4

Unit - III

5. (a) Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous, bounded function. Show that there exists $x_0 \in \mathbb{R}$ such that $f(x_0) = x_0$ 4

(b) Prove or disprove : There exists a continuous function $f : \mathbb{R} \to \mathbb{R}$ such that $f(\mathbb{Q}) \subset \mathbb{R} \setminus \mathbb{Q}$ and $f(\mathbb{R}\setminus\mathbb{Q}) \subset \mathbb{Q}$ 5

(c) Let $f : \mathbb{R} \to \mathbb{R}$ be a monotone function such that f satisfies the intermediate value property. Show that f is continuous. 5

6. (a) Show that $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} \sin\frac{1}{x} & if \quad x \neq 0 \\ 0 & if \quad x = 0 \end{cases}$$

satisfies the intermediate value property. 5

(b) Prove or disprove : There exists a continuous function $f : (0, 1) \to \mathbb{R}$ such that f attains its both supremum and infimum in (0, 1). 3

(c) Let $f : [0, 1] \to \mathbb{R}$ and $g : [0, 1] \to \mathbb{R}$ be continuous functions such that $f(x) < g(x)$ for all $x \in [0, 1]$. Prove that $\sup_{x \in [0,1]} f(x) < \sup_{x \in [0,1]} g(x)$ 3

(d) Define an unbounded function. Is every continuous function $f : \mathbb{R} \to \mathbb{R}$ bounded? Justify. 1+2=3

Unit - IV

7. (a) State and prove Rolle's theorem. 1+3=4

(b) Prove or disprove : If $f : \mathbb{R} \to \mathbb{R}$ is a differentiable function such that $f'(x_0) \neq 0$ for some $x_0 \in \mathbb{R}$, then there exists $\delta > 0$ such that f is one-one in $(x_0 - \delta, x_0 + \delta)$ 4

(c) Give an example of a function $f : \mathbb{R} \to \mathbb{R}$, with justification, such that f has a local minimum at some $x_0 \in \mathbb{R}$, but f is not differentiable at x_0. 2

(d) Show that every continuous function $f : [0,1] \to \mathbb{R}$ is Riemann integrable. 4