PG (CBCS) EVEN SEMESTER EXAMINATION, 2023

MATHEMATICS

4th Semester

Course No. : MTMCC - 401

(Functional Analysis)

Full Marks : 70

Pass Marks : 28

Time : 3 hours

The figures in the margin indicate full marks for the questions

(Answer one question from each unit)

Unit - I

1. (a) Show that $(C_{00}, \|.\|_\infty)$ is not complete. 5

(b) Let Y be a closed subspace of a normed linear space X and let $\pi(x) = x + Y \ \forall\, x \in X$. If $x_0 \in X$ and r>0, show that 5

$$\pi\,(B_r\,(x_0)) = B_r\,(\pi\,(x_0))$$

(c) Show that the norms $\|.\|_2$ *and* $\|.\|_\infty$ are not equivalent on l^2. 4

2. (a) Show that in the definition of $\|.\|_\infty$, supremum can be replaced by maximum in c_0 but not in c. 5

(b) Let Y be a closed subspace of a normed linear

space X. Show that $\|.\|_0$ on X/Y given by
$\| x + Y \|_0 = \inf \{ \| x + y \| : y \in Y \}$
is a norm on X/Y. Is $\|.\|_0$ as defined above a norm on X/Y if Y is not closed in X? Justify. 4+2=6

(c) Show that S_X is not compact if X is an infinite dimensional normed linear space. 3

Unit - II

3. (a) Let $f((x_n)) = \sum_{n=1}^{\infty} x_n \ \forall (x_n) \in l^1$

Show that f is linear. Also show that $f: (l^1, \|.\|_1) \to \mathbb{K}$ is bounded with $\| f \| = 1$, whereas $f: (l^1, \|.\|_\infty) \to \mathbb{K}$ is not bounded. 1+5=6

(b) Show that every closed subspace of a reflexive space is reflexive. 4

(c) Prove or disprove : If Y is a non-zero proper subspace of a normed linear space X, then each $f \in X^*$ is a Hahn-Banach extension of $f/_Y \in Y^*$ to X. 4

4. (a) Prove or disprove : For every infinite dimensional Banach space X, there exists a linear map $T : X \to Y$ such that ker T is not closed in X. 4

(b) State Hahn-Banach theorem. Let X be a normed linear space and let $x_0 \in X$ be such that $f(x_0) = 0 \ \forall f \in X^*$. Show that $x_0 = o$. 1+4=5

(b) Let H be a Hilbert space. Show that $T \in \mathcal{B}(H)$ is normal iff
$\langle Tx, Ty \rangle = \langle Tx^*, Ty^* \rangle \quad \forall x\, y \in H$ 4

(c) If $T(x) = f(x)\, y_0$ for all $x \in X$, where X is a normed linear space, $y_0 \in X$ and $f \in X^*$, then show that T is a compact operator. 5

10. (a) Let X, Y be Banach spaces and let $T \in B_0(X, Y)$. If Y is infinite - dimensional, then show that T (X) is not closed in Y. 5

(b) Let H be a Hilbert space and let $T, S \in B(H)$. Show that TS = 0 iff ran$T \perp$ ran S. 5

(c) Let H be a Hilbert space. If $T \in B(H)$ and if $\alpha, \beta \in \mathbb{K}$ such that $|\alpha| = |\beta|$, then show that $\alpha T + \beta T^*$ is normal. 4

★★★

(b) Let H be a Hilbert space and let C be a nonempty, closed, convex subset of H. Show that for each $x \in H \setminus C$, there exists unique $y \in C$ such that $\| x - y \| = \text{dist}\,(x, C)$ 5

(c) State and prove Riesz representation theorem. 1+4=5

8. (a) Let X be a normed linear space and let $y \in X$. Show that $f : X \to \mathbb{K}$ defined by $f(x) = \langle x, y \rangle \quad \forall x \in X$ is a bounded linear functional with $\| f \| = \| y \|$. 4

(b) Let M be a closed subspace of a Hilbert space H. Show that $M^{\perp\perp} = M$. Also, show that if M is any subspace of H, then $M^{\perp\perp\perp} = M^{\perp}$ 5

(c) Let H be an infinite dimensional Hilbert space. Show that no orthonormal basis of H can be Hamel basis for H. 5

Unit - V

9. (a) Define an orthogonal projection on a closed subspace of a Hilbert space H.
Show that for any orthogonal projection $P_M : H \to H$ on a closed subspace M of a Hilbert space.
(i) $\ker P_M = M^{\perp}$
(ii) $\operatorname{ran} P_M = M$

(c) Let X be a normed linear space. Show that X^* is separable only if X is separable. 5

Unit - III

5. (a) Show that a Banach space can not have a countably infinite Hamel basis. 4

(b) State and prove Banach-Steinhauss theorem. 1+4=5

(c) Let X and Y be normed linear spaces and let $T : X \to Y$ be open, continuous and linear. Show that if X is complete, then Y is complete. 5

6. (a) Let $T(x) = x + x'$ for all $x \in c^1[0, 1]$. Prove that $T : (C^1[0,1], \|.\|_\infty) \to (C[0,1], \|.\|_\infty)$ is a discontinuous linear map with a closed graph. 5

(b) State and prove uniform boundedness principle. 1+4=5

(c) Prove or disprove : If X is a normed linear space and $f : X \to \mathbb{K}$ is any linear functional, then graph of f is closed. 4

Unit - IV

7. (a) Let $(X, <,>)$ be an inner product space and let $x, y \in X$. Show that $x \perp y$ iff $\| x + \alpha y \| = \| x - \alpha y \| \quad \forall \alpha \in \mathbb{K}$ where $\|.\|$ on X is induced by the inner product $<,>$. 4