(c) Assume that m (E) < $\alpha$. Let $\langle f_n \rangle$ be a sequence of function which is uniformly integrable over E. If $f_n \to f$ pointwise a.e. on E, then show that $f$ is integrable over E. 4

10. (a) State and prove Boundedness Convergence theorem. 1+5=6

(b) Let f be a non-negative measurable function defined on a measurable set E. For any c> 0 show that 4

$$m(\{x \in E : f(x) \geq c\}) \leq \frac{1}{c}\int_E f \, dm$$

(c) Assume that m (E) < $\propto$. Let $\langle f_n \rangle$ be a sequence of measurable function on E that converges pointwise a.e. on E to $f$ and $f$ is finite valued a.e. on E. Show that $f_n \to f$ in measure on E. 4

★★★





# PG (CBCS) EVEN SEMESTER EXAMINATION, 2023

## MATHEMATICS

4th Semester

Course No. : MTMCC - 402

**( Real Analysis II )**

Full Marks : 70

Pass Marks : 28

Time : 3 hours

*The figures in the margin indicate full marks for the questions*

(Answer one question from each unit)

Unit - I

1. (a) Let f : [a, b] → R be any bounded function. Show that f is Riemann integral over [a, b] with $\int_a^b f(x)\,dx = 1$ if and only if there exists a sequence of partitions $\langle P_n \rangle$ of $[a, b]$ such that $U(P_n, f) \to I$ and $L(P_n, f) \to I$ as $n \to \infty$. 6

(b) Let $\langle f_n \rangle$ be a sequence of differentiable functions in [a,b] such that $f_n \to f$ uniformly in [a,b] and $f_n' \to g$ uniformly on [a, b] for some suitable functions $f$ and g on [a, b]. Show that f is differentiable in [a, b] and $f' = g$ on [a, b] 5

( Turn Over )

(c) Let $f : [a, b] \to R$ be a bounded function such that U (P, f )=L (P, f ) for some partition P of [a, b] Show that f must be constant on a[, b]. 3

2. (a) Show that every continuous function on a closed and bounded interval is Riemann integrable. 4

(b) Let $f : [a, b] \to \mathbb{R}$ be continuous and assume that f keeps the same sign throughout [a, b].

Show that $\int_a^b f(x)\,dx = 0$ if and only if $f = \hat{0}$ 4

(c) Let $f : [-1,1] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 1 & \text{if } x = 1/n \text{ for some } n \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

Evaluate the Riemann integral $\int_{-1}^{1} f(x)\,dx$, if it exists. Further if

$$F(x) = \int_{-1}^{x} f(t)\,dt , \qquad x \in [-1, 1]$$

then show that F is differentiable at 0 with $F'(0) = f(0)$, eventhough f is discontinuous at 0. 6

## Unit - II

3. (a) State the two forms of fundamental theorem of calculus and provle any one of them. 2+4=6



that f is Lebesgue measurable iff $f^{-1}(B)$ is a measurable set for every Borel subset B of $\mathbb{R}$. 5

(b) Prove or disprove : Composition of two measurable functions is again measurable. 5

(c) Suppose a function $f : E \to R$ has a measurable domain E and $f$ is continuous except at a finity number of points of E. Is $f$ necessarily measurable? Justify your answer. 4

8. (a) Let f and g be measurable functions. Show that $f+g$ and $fg$ are also measurable. 3+3=6

(b) Show that every measurable function defined on a measurable domain E can be expressed as a poinwise limit of a sequence of simple functions on E. 8

## Unit - V

9. (a) Let $f$ be a bounded function defined on a measurable domain E of finite measure. Show that $f$ is Lebesgue integrable over E. 5

(b) State and prove Fatou's lemma. Hence or otherwise, deduce Monotone Convergence theorem. 1+3+1=5

## Unit - III

5. (a) For each $a \in R$, show that $(a, \propto)$ is Legesgue measurable. Deduce that any interval of $\mathbb{R}$ is measurable. 3+3=6

(b) Show that Cantor set is uncountable and have Legesgue measure 0. 5

(c) Let $A, B \subseteq R$. Assume that there exists a measurable set G of $\mathbb{R}$ such that $A \subseteq G$ and $B \cap G = \varphi$. Show that
$m^* (A \cup B) = m^* (A) + m^* (B)$ 3

6. (a) Show that Lebesgue measure is countably additive on the set of all measruable subsets of $\mathbb{R}$. 6

(b) Show that a strictly increasing continuous map $f$ from $\mathbb{R}$ onto $\mathbb{R}$ takes Borel subsets of $\mathbb{R}$ onto Borel subsets of $\mathbb{R}$. 5

(c) Let A be any measurable subset of $\mathbb{R}$ such that $m(A) > 0$. Show that there exists $x, y \in A$ such that $x - y \in \mathbb{R} \setminus \mathbb{Q}$. 3

## Unit - IV

7. (a) Let $f : E \to \mathbb{R}_e$ be any extended real valued function defined on a measurable set $E \subseteq \mathbb{R}$. Show



(b) Show that Lebesgue outer measure is monotone but not strictly monotone. 2+2=4

(c) Show that union of two measurable sets is measurable. 4

4. (a) Let $\langle f_n \rangle$ be a sequence of Riemann integrable functions defined on [a,b] such that $f_n \to f$ uniformly in [a,b]. Show that $f$ is Riemann integrable in [a,b] and that 6

$$\lim_{n \to \infty} \int_a^b f_n(x)\, dx = \int_a^b f(x)\, dx$$

(b) Let $f : [1, 2] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 1 & if \quad x = 1 \\ 0 & if \quad 1 \le x \le 2 \end{cases}$$

and $F(x) = \int_1^x f(t)\, dt, \qquad x \in [1, 2]$

Evaluate the Riemann integral $\int_1^2 f(x)\, dx$ Further show that F is differential at 1 but $F'(1) \neq f(1)$. Does this contradict fundamental theorem of calculus? Justify your answer. 5

(c) Show that Lebesgue outer measure in countably subadditive. 3