8. (a) Show that continuous image of a compact topological space is compact. 5

   (b) Show that a subset E of $\mathbb{R}$ with the usual topology is connected iff E is an interval. 5

   (c) Let (X, τ_1) and (Y, τ_2) be topological spaces. Show that $\{G \times H : G \in \tau_1 , H \in \tau_2 \}$ is a basis for some topology on X x Y. 4

Unit - V

9. (a) Define a Hausdoff space. Show that every Hausdorff space is a T_1-space. Is the converse true? Justify. 1+2+3=6

   (b) Show that every second countable space is both first countable and separable. Is a separable space necessary first countable? Justify. 3+3+2=8

10. (a) Define a normal space. Show that $\mathbb{R}$ with the usual topology is normal. 1+4=5

    (b) State Urysohn's lemma on normal spaces and prove it for metric spaces. 1+4=5

    (c) Prove or disprove : $\mathbb{R}$ with the discrete topology is first countable but not second countable. 4

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PG (CBCS) EVEN SEMESTER EXAMINATION, 2023

MATHEMATICS

2nd Semester

Course No. : MTMCC - 201

(Topology)

Full Marks : 70

Pass Marks : 28

Time : 3 hours

The figures in the margin indicate full marks for the questions

(Answer one question from each unit)

Unit - I

1. (a) Show that the function
   $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$
   defined by $d(x,y) = \begin{cases} 5 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$
   is a metri on $\mathbb{R}$. 4

   (b) Let d denote the discrete metric on $\mathbb{R}$. Show that a sequence (x_n) in $\mathbb{R}$ converges iff there exist $x_0 \in \mathbb{R}$ and $n_0 \in \mathbb{N}$ such that $\forall\ n \geq n_0,\ x_n = x_0$ 5

   (c) Show that every finite subset of any metric space (X, d) is closed in (X, d). Is the converse true? Justify. 3+2=5

2. (a) Consider $\mathbb{R}$ with the usual metric. Show that $\mathbb{N}$ is closed in $\mathbb{R}$. 4

   (b) Let A, B be two subsets of a metric space X. Show that $\overline{A \cup B} = \bar{A} \cup \bar{B}$ 5

   (c) Let (X, d_1) and (y, d_2) be metric spaces. Show that $f : (X, d_1) \to (Y, d_2)$ is continuous iff $f^{-1}(G)$ is closed in (X, d_1) for each closed set G in (Y, d_2). 5

Unit - II

3. (a) Define a Cauchy sequence in a metric space. Show that every convergent sequence in a metric space is a Cauchy sequence. Is the converse true? Justify. 1+2+2=5

   (b) State and prove Cantor's intersection theorem. 1+4=5

   (c) Define a nowhere dense set in a metric space. Show that a set A in a metric space (X , d) is nowhere dense iff $\bar{A}$ is nowhere dense. 1+3=4

4. (a) Show that every complete subspace of a metric space X is closed in X. Is the converse true? Justify. 3+3=6

   (b) State and prove Baire's category theorem. 1+4=5

   (c) Show that (0, 1)is not compact in $\mathbb{R}$ with the usual metric. 3

Unit - III

5. (a) Let τ_1 and τ_2 denote respectively cofinite and usual topologies on $\mathbb{R}$. Show that $\tau_1 \subset \tau_2$ 4

   (b) Let (X, d) be a metric space and let $\tau = \{ G \subseteq X : G$ is d-open$\}$ Show that τ is a topology on $\mathbb{R}$. 5

   (c) Let (X, τ) and (Y, σ) be topological spaces. Show that a function $f : X \to Y$ is continuous iff for each subset A, $f(\bar{A}) \subset \overline{f(A)}$ 5

6. (a) Show that $\tau = \mathbb{R} \cup \{ G \subset \mathbb{R} : 0 \notin G \}$ is a topology on $\mathbb{R}$. 5

   (b) Let $\{ \tau_\alpha : \alpha \in \Lambda \}$ be any family of topologies on X. Show that $\bigcap_{\alpha \in \Lambda} \tau_\alpha$ is a topology on $\mathbb{R}$. 5

   (c) Define homeomorphism between metric spaces. Show that $\mathbb{R}$ with the usual topology is homeomorphic to (0, 1) with the subspace topology. 4

Unit - IV

7. (a) Show that $\mathbb{R}$ with the discrete topology is not compact. 4

   (b) Show that a topological space (X, τ) is connected iff the only clopen subsets of X are X and ϕ. 5

   (c) Let τ_u denote the usual topology on $\mathbb{R}$. Show that the product topology $\tau_u \times \tau_u$ on $\mathbb{R}^2$ is the usual topology on $\mathbb{R}^2$. 5