(b) State and prove Eisenstein's criterion for irreducibility over $\mathbb{Q}$. 6

(c) Show that the polynomial $f(X) = 1 + pX + p^{n-1} X^n$, where p is any prime, is irreducible over $\mathbb{Q}$. 3

Unit - V

9. (a) Let $F \subseteq E \subseteq K$ be a tower of fields. Show that $[K : F] = [K : E] [E : F]$. 5

(b) Let $K|F$ be a field extension and $a \in K$ be algebraic over F. Show that there exists a unique menic irreducible polynomial $p(X) \in F[X]$ such that $p(a) = 0$. 6

(c) Prove that every finite extension is a algebraic extension. 3

10. (a) State and prove Kronecker's theorem. 6

(b) Let $K|F$ be a field extension. Show that the set $\overline{F}$, of all elements of K which are algebric over F, is an intermediate subfield of K. 4

(c) Show that there are infinitely many intermediate fields between $\mathbb{Q}$ and $\mathbb{R}$. 4

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PG (CBCS) EVEN SEMESTER EXAMINATION, 2023

MATHEMATICS

2nd Semester

Course No. : MTMCC - 202

(Algebra - II)

Full Marks : 70

Pass Marks : 28

Time : 3 hours

The figures in the margin indicate full marks for the questions

(Answer one question from each unit)

Unit - I

1. (a) Write down the units and zero-divisor in (i) $\mathbb{Z}_n$, (ii) $M_n(\mathbb{R})$, and (iii) R [X] where R is an integral domain. 4

(b) Show that the set $\mathbb{Z}[\sqrt{m}] = \{a + b\sqrt{m} / a, b \in \mathbb{Z}\}$ is an integral domain. 5

(c) Let R be an integral domain and let $f(x)$ and $g(x) \in R[X]$ with $f(x) \neq 0$, $g(x) \neq 0$. Show that $\deg(f(x) \cdot g(x)) = \deg(f(x)) + \deg(g(x))$. 5

2. (a) Show that the characteristic of an integral doman R must be either zero or a prime. 4

(b) Let I be a finite integral domain. Show that the order of I is equals to p^n ; where p is some prime number and $n \in \mathbb{N}$. 5

(c) Let R be any commutative ring with unity and let I be any ideal of R. Show that there is a one-to-one correspondence between the ideals of R/I and the ideals of R containing I. 5

Unit - II

3. (a) Fine all the units of
$\mathbb{Z}[\sqrt{-m}] = \{a + b\sqrt{-m}\,;\, a, b \in \mathbb{Z}\}$ for $m \geq 1$. 5

(b) Let R be a Commutative Ring with unity and $a \in R$ is a unit and $b \in R$ is a nil potent. Show that a+b and a-b are units. 6

(c) Find all the units and zero divisors in $\mathbb{Z}_{12}$. 3

4. (a) Let R be a commutative ring with unity. Show that the binomial theorem for positive integral index holds in R. 7

(b) Let $C(\mathbb{R}, \mathbb{R})$ be the ring of all continuous real-valued functions on $\mathbb{R}$. Find all the units and zero divisors in $C(\mathbb{R}, \mathbb{R})$. Is every element of $C(\mathbb{R}, \mathbb{R})$ either a unit or a zero divisor? Justify? 7

Unit - III

5. (a) Let R be a commutative ring with unity and let M be an ideal of R. Show that M is a Maximal ideal iff R/M is a field. 5

(b) Show that $\mathbb{Z}$ [X] is not a principal ideal domain. 4

(c) Show that the ring of all continuous real-valued functions C ([0, 1], $\mathbb{R}$) is not a field. Further, show that for each fixed $0 \leq r \leq 1$ the set $M := \{f \in C([0, 1], \mathbb{R}) : f(r) = 0\}$ is a Maximal ideal of C ([0, 1], $\mathbb{R}$) 2+3=5

6. (a) Show that $\mathbb{Z}$ ($\sqrt{-5}$) is not a unique factorization domain. 4

(b) Show that in a UFD, every irreducible element is a prime element. 5

(c) In a PID, show that every non-zero, non-unit element is divisible by an irreducible element. 5

Unit - IV

7. (a) State and prove Gauss Lemma. 1+5=6

(b) Let R be a UFD. Show that R [X] is also a UFD. 8

8. (a) Let F be a field and $f(X) \in F[X]$ be any polynomial of degree $n \geq 2$ has a root in F. Show that f (X) is irreducible over F. Is the converce true? Explain. 4+1=5