(b) Define path-connectedness in a topological space. Show that every path connected space is connected. Is the converse true? Justify. 1+4+2=7

Unit - V

9. (a) Show that every second countable space is first countable. Is the converse true? Justify. 3+4=7

(b) Show that a topological space (X, τ) is a T_1-space if and only if $\{x\}$ is closed in (X, τ) for each $x \in X$. 5

(c) Show that the real line is Hausdoff. 2

10. (a) Define a normal space. Show that every T_4 - space is a T_3 - space. 1+5=6

(b) State Urysohn's lemma and prove it for metric spaces. 2+6=8

★★★

PG (CBCS) Even Semester Examination, May 2025

MATHEMATICS

2nd Semester

Course No. : MTMCC - 201 (O)

(Topology)

Full Marks : 70

Pass Marks : 28

Time : 3 hours

The figures in the margin indicate full marks for the questions

(Answer one question from each unit)

Unit - I

1. (a) Define an open ball and an open set in a metric space. 2+4+1=7
Show that in a metric space, every open ball is an open set. Is the converse true? Justify.

(b) Define limit points of a set in a metric space. Show that in any metric space (X, d), if $A \subset X$, then the set $A \cup A'$ is always closed in X, where A' is the set of all limit points of A in X. 1+6=7

2. (a) Let (X, d) be a metric space and let A be a subset of X. Show that A^0, the interior of A is an open subset of A. Also, show that if G is open and $G \subset A$, then $G \subset A^0$. 4+3=7

(Turn Over)

(b) Define continuity of a function from a metric space (X, d_1) into a metric space (Y, d_2) at a point $x_0 \in X$. 1+4+2=7

Show that $f : (X, d_1) \to (Y, d_2)$ is continuous if and only if $f(\bar{A}) \subset \overline{f(A)} \quad \forall A \subset X$. Give an example to show that the inclusion can be strict.

Unit - II

3. (a) Define a complete metric space. State and prove Cantor's intersection theorem. 1+1+5=7

(b) Show that in the usual metric space $(\mathbb{R}, d)$, $A \subset \mathbb{R}$ is compact if and only if A is closed and bounded. 7

4. (a) Define a dense set in a metric space. Show that countable intersection of open and dense sets in a metric X is dense in X. 1+6=7

(b) Let (X, d) be a compact metric space and let (Y, p) be any metric space. If $f : X \to Y$ is a continuous surjection, then Y is also compact. 5

(c) Show that $\mathbb{R}$ with discrete metric is not compact. 2

Unit - III

5. (a) Let X be a nonempty set and let $\{\tau_\alpha : \alpha \in \Lambda\}$ be any family of topologies on X. Show that $\bigcap_{\alpha \in \Lambda} \tau_\alpha$ is a topology on X. Is $\bigcup_{\alpha \in \Lambda} \tau_\alpha$ a topology on X? Justify.

4+3=7

(b) Let (X, d) be a metric space. Show that $\tau_d = \{ G \subset X : \forall x \in G, \exists r > 0 \text{ such that } B_r(x) \subset G \}$ is a topology on X.

If τ is any topology on X, is it necessary that there exists a metric d on X such that $\tau = \tau d$? Justify. 4+3=7

6. (a) Show that

$\tau = \{ G \subset \mathbb{R} : \mathbb{R} \setminus G \} \text{ is finite } \} \cup \{\phi\}$ is a topology on $\mathbb{R}$. 4+3=7

If τ_u represents the usual topology on $\mathbb{R}$ then show that $\tau u \tau_u$ is a topology on $\mathbb{R}$.

(b) Let (X, τ) be a topological space and let A be a subset of X. Show that $A^0 \in \tau$ and that

$A^0 = X \setminus \overline{(X \setminus A)}$ 2+5=7

Unit - IV

7. (a) State and prove Tychonoff's theorem. 1+8=9

(b) Show that if A is a connected subset of the real line, then A is an interval. 5

8. (a) Let (X, τ) be a Husdorff space and let A be a compact subset of X. Show that A is closed in X. 7