PG Even Semester Examination, May 2025

MATHEMATICS

2nd Semester

Course No. : MAT - 551

(Basic Analysis)

Full Marks : 70

Pass Marks : 28

Time : 3 hours

The figures in the margin indicate full marks for the questions

(Answer one question from each unit)

Unit - I

1. (a) When is a function said to be continuous at $c \in [a, b]$? Show that f is continuous at $x = c \in [a,b]$ iff for every sequence $(x_n) \subset [a, b]$ converging to c, we have $f(x_n) \to f(c)$ as $n \to \propto$. 1+4=5

   (b) State and prove intermediate value theorem. 1+5=6

   (c) Does there exist a monotone functin $f: \mathbb{R} \to \mathbb{R}$ which is continuous exactly at the rational numbers? Justify your answer. 3

2. (a) Construct a function $f: \mathbb{R} \to \mathbb{R}$ which is continuous exactly at the irrational numbers. 5

(Turn Over)

(b) Show that the set of points of discontinuities of a monotone function $f: \mathbb{R} \to \mathbb{R}$ is countable. 6

(c) Let E be a bounded subset of $\mathbb{R}$ which is not closed. Give an example of a function $f: E \to \mathbb{R}$ which is continuous in E but not bounded in E, with proper justification. 3

Unit - II

3. (a) State and prove Rolle's theorem. 1+4=5

(b) Let $f : [a, b] \to \mathbb{R}$ be continuous. Show that f is Riemann integrable. Is the converse true? Justify. 3+3=6

(c) Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable such that $f'(x_0) = 0$ for some $x_0 \in \mathbb{R}$ Is it necessary that there always exist a, b $\in \mathbb{R}$ such that $a < x_0 < b$ and $f(a) = f(b)$? Justify your answer. 3

4. (a) Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable. Show that $f' : \mathbb{R} \to \mathbb{R}$ possesses the intermediate value property. Is f' necessarily continuous? Justify. 3+2=5

(b) State and prove inverse function theorem for a differentiable function in $\mathbb{R}$. 1+4=5

(c) Let $f : [a, b] \to \mathbb{R}$, $g : [a, b] \to \mathbb{R}$ be bounded functions such that $\{ x \in [a, b] : f(x) \neq g(x)\}$ is at most finite. If f is Riemann integrable in [a, b],

(b) Let (X, d) be a metric space and let A be a non empty subset of X. Show that $f : X \to \mathbb{R}$ given by $f(x) = \text{dist}(x, A)\ \forall\ x \in X$ is uniformly continuous. 4

(c) Let (X, d) be a discrete metric space and let $A \subset X$. Show that A is compact iff A is finite. 4

10. (a) State and prove Baire category theorem. 1+5=6

(b) Let (X, d_1) and (Y, d_2) be metric spaces and let $f : X \to Y$ be a function. Show that f is continuous iff

$$f(A)^0 \subset f(A^0)\ \forall\ A \subset X$$ 5

(c) Show that $\mathbb{R}$ with the discrete metric is disconnected. 3

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then show that g is also Riemann integrable in [a, b] and $\int_a^b f(x)\,dx = \int_a^b g(x)\,dx$ 4

Unit - III

5. (a) Let $f_n(x) = x^n$ for all $x \in [0, 1]$ and for all $n \in \mathbb{N}$. Show that (fn) converges pointwise to a suitable function f in [0, 1]. Is the convergence uniform? Justify your answer. 2+2=4

(b) Let (f_n) be a sequence of continuous functions defined on [a, b] such that $f_n \to f$ uniformly on [a, b]. Show that f must be continuous on [a, b]. Is the conclusion holds if the convergence is only pointwise? Justify. 4+2=6

(c) Let $f : [a, b] \to \mathbb{R}$ be differentiable with $|f'(x)| \leq M$ $\forall\, x \in [a, b]$ and for some M>0. Show that f is of bounded variation in [a, b]. 4

6. (a) Let $f_n, f : [a, b] \to \mathbb{R}$ be functions $f_n \to f$ pointwise & let $M_n = \sup\{|f_n(x) - f(x)| : x \in [a, b]\} \in \mathbb{R}\ \forall\, n \in \mathbb{N}$ show that $f_n \to f$ uniformly iff $M_n \to 0$ as $n \to \infty$ 5

(b) Let $f_n \to f$ uniformly in [a, b] and that f_n are differentiable in (a, b). Is it necessary that f is differentiable in (a, b)? If f is differentiable in (a, b), is it necessary that $f_n' \to f'$ uniformly? Justify. 3+3=6

(Turn Over)

(c) Construct a continuous function in some interval [a, b] which is not of bounded variation in [a, b]. 3

Unit - IV

7. (a) Show that in a metric space, every closed ball is a closed set. 4

(b) Let (X, d) be a metric space and let A ($\neq \phi$) be a subset of X. Show that 5

$$\bar{A} = \{x \in X : \text{dist}(x, A) = 0\}$$

(c) State and prove cantor's intersection theorem. 1+4=5

8. (a) Show that in a metric space every singleton is a closed set. 3

(b) Let (X, d) be a metric space and let A be a subset of X. Show that A is closed in X iff A contains all its limit points. 5

(c) Define a complete metric space. Is $\mathbb{R}$ with the discrete metric complete? Justify. 1+5=6

Unit - V

9. (a) Let (X, d) be a metric space and let A be a subset of X. Show that A is dense in X if and only if $\bar{A} = X$. 6