(b) State and prove Vitali convergence theorem. 1+4=5

(c) Let $\langle fn \rangle$ be a sequence of Lebesgue measurable functions from $\mathbb{R}$ to $\mathbb{R}$ such that $fn \to f$ in measure, for some measurable function $f : \mathbb{R} \to \mathbb{R}$ show that there exists a subsequence $\langle fn_k \rangle$ of $\langle fn \rangle$ such that pointwise a.e. in $\mathbb{R}$. 5

10. (a) Let $\langle fn \rangle$ be a sequence of non-negative integrable function on a measurable set $\phi \neq E \subseteq \mathbb{R}$. Suppose $fn \to 0$ pointwise a.e. on E. Show that $\lim_{n\to\infty} \int_E f\, dm = 0$ iff $\langle fn \rangle$ is uniformly integrable and tight over E. 6

(b) Let $f : \mathbb{R} \to [-\infty + \infty]$ be integrable and $\left| \int_R f\, dm \right| = \int_R |f|\, dm$. Show that $f \geq 0$ a.e. on $\mathbb{R}$ or $f \leq 0$ a.e. on $\mathbb{R}$ 5

(c) Consider $f_n : \mathbb{R} \to \mathbb{R}$, $f_n i : \chi_{[n+\infty]}$ $\forall$ n $\geq$ 1. Show that $\langle fn \rangle$ does not converge to the zero function in measure. 3

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PG Even Semester Examination, May 2025

MATHEMATICS

4th Semester

Course No. : MAT - 651

(Advanced Analysis)

Full Marks : 70

Pass Marks : 28

Time : 3 hours

The figures in the margin indicate full marks for the questions

(Answer one question from each unit)

Unit - I

1. (a) Let $f: \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x,y) = \begin{cases} \dfrac{x^\alpha\, y^\beta}{x^2 + xy + y^2} & when \quad (x,y) \neq (0,0) \\ 0 & when \quad (x,y) = (0,0) \end{cases}$$

show that f is continuous in $\mathbb{R}^2$ iff $\alpha + \beta > 2$ 5

(b) Give example, with justification of a function $f: \mathbb{R}^2 \to \mathbb{R}$ which has same directional derivative at (0, 0) along every direction but is discontinuous at (0, 0). 4

(c) Let Ω be any open subset of $\mathbb{R}^n$. If $f : \Omega \to \mathbb{R}^m$ is

differentiable at $a \in \Omega$, show that the derivative of f at a is uniquely determined. 5

2. (a) If Ω is open in $\mathbb{R}^n$ and if $f : \Omega \to \mathbb{R}^m$ is differentiable at $a \in \Omega$, show that f is continuous at a. Is the converse true? Justify your answer. 2+2=4

(b) Let $f : \mathbb{R}^n \to \mathbb{R}$ be differentiable at $a \in \mathbb{R}^n$ and let $u \in \mathbb{R}^n$ be any non-zero vector show that 5

$$D f(a)(u) = \nabla f(a).\ u$$

(c) Let $f : \mathbb{R}^n \to \mathbb{R}$ be defined $f(x) = ||x||^2 \quad \forall\ x \in \mathbb{R}^n$ show that f is differentiable in $\mathbb{R}^n$. 5

Unit - II

3. (a) Let $f : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^m$ be differentiabl at $a \in \Omega$ where Ω is open in $\mathbb{R}^n$. Let $f = (f_1, f_2, \ldots f_m)$ where $f_i : \mathbb{R}^n \to \mathbb{R}$ for each i, $1 \le i \le m$. Show that $\frac{\partial f_i}{\partial x_j}$ exists for each $1 \le i \le m$ and $1 \le j \le n$ at Q. Also find the matrix representation of the derivative map $Df(a)$. 5

(b) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be differentiable and let $g(x) = f\big(x, f(x,x)\big)$ for each $x \in \mathbb{R}$. If $f(1, 1) = 1$, $f_x(1, 1) = 2$ and $f_y(1, 1) = 3$, then find $g'(1)$ 4

(c) State inverse function theorem. Determine all points in $\mathbb{R}^2$ where the function , $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x, y) := (x + y, x^2 + y)$ is locally invertible. 2+3=5

(c) Consider a sequence $\langle f_n \rangle$, $f_n : \chi_{[n, n+1]} \ \forall \ge 1$. Find the pointwise limit $f : \mathbb{R} \to \mathbb{R}$ of the sequence $\langle f_n \rangle$. Show further that 2+2=4

$$\lim_{n\to\infty} \int_{\mathbb{R}} f_n\, dm \neq \int_{\mathbb{R}} \left(\lim_{n\to\infty} f_n\right) dm$$

8. (a) State and prove Lebesgue monotone convergence theorem. 1+5=6

(b) Let E be a non-empty Lebesgue measurable subset of $\mathbb{R}$ having finite Lebesgue measure. Let $\langle f_n \rangle$ be a uniformly bounded sequence of Lebesgue measurable functions on E. If $\lim_{n\to\infty} f_n(x) = f(x)$ for almost all $x \in E$ for some Lebesgue measurable function f on E, show that 5

$$\lim_{n\to\infty} \int_E f\, dm = \int_E f\, dm < +\infty$$

(c) Using Lebesgue dominated convergence theorem, evaluate 3

$$\lim_{n\to\infty} \int_{-2}^{2} \frac{x^{2n}}{1 + x^{2n}}\, dx$$

Unit - V

9. (a) Let $f : \mathbb{R} \to [-\infty + \infty]$ be Lebesgue integrable and let $E_n : \{ x \in \mathbb{R} : f(x) \ge n \}$ for all $n \in \mathbb{N}$. Show that $\lim_{n\to\infty} \big(n, m(E_n)\big) = 0$ 4

then find $m^*(A)$ where

$$A = \left\{ x \in [0,2] ;\ f(x) \in \left(\frac{9}{10}, \frac{5}{4}\right)\right\}$$

(c) Show that $f : \mathbb{R} \to \mathbb{R}$ is Lebesgue measurable iff $f^{-1}(B)$ is Lebesgue measurable for each Borel Subset of $\mathbb{R}$. 5

6. (a) Give an example, with justification, of an unbounded subset A of $\mathbb{R}$ such that $m^*(A) = 4$ 3

(b) Let $\langle d_n \rangle$ be a sequence of pairwise disjoint measurable subsets of $\mathbb{R}$. Show that 5

$$m^4\left(\bigcup_{r=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} m^4(E_n)$$

(c) Define a Lebesgue measurable function. Show that if $f : \mathbb{R} \to \mathbb{R}$ is continuous, then f is Lebesgue measurable. Is the converse true? Justify. 1+2+3=6

Unit - IV

7. (a) Define $f : \mathbb{R} \to [0 + \infty]$ be a bounded Lebesgue measurable function. Show that there exists an increasing sequence $\langle \varphi_n \rangle$ of non-negative simple measurable funcions on $\mathbb{R}$ such that $\varphi_n \to f$ uniformly in $\mathbb{R}$. 6

4

(b) State and prove Fatou's lemma. 1+3=4

4. (a) State and prove chain rule of differentiation in multivariable calculus. 1+5=6

(b) Consider defined by 2+2=4

$$f(x,y) := \begin{cases} (x^2+y^2)\ \sin\left(\dfrac{1}{\sqrt{x^2+y^2}}\right) & when\ \ (x,y) \neq (0,0) \\ 0 & when\ \ (x,y) = (0,0) \end{cases}$$

Examine if f is differentiable at (0, 0). Is f_x continuous at (0, 0)? Justify your answer.

(c) Show that there are points (x_0, y_0, u_0, v_0) in $\mathbb{R}^4$ which satisfy the equation

$x - e^u \cos v = 0$; $v - e^y \sin x = 0$

Around any such point, show that there exists a unique solution $(u, v) = \varphi(x, y)$ satisfying $det\ [D\varphi(x,y)] = \dfrac{\theta}{x}$ 4

Unit - III

5. (a) Show that B ($\mathbb{R}$), the Borel σ-algebra on $\mathbb{R}$ is generated by the class of all closed subsets of $\mathbb{R}$. 5

(b) If $g : [0, 2] \to \mathbb{R}$ is defined by 4

$$g(x) = \begin{cases} x^2 & if\ \ 0 \leq x \leq 1 \\ 3 - x & if\ \ 1 < x < 2 \end{cases}$$