(c) If $(X, \|.\|)$ is a unital Banach algebra with $e(\neq 0)$ being the unit element, show that $\| e \| \geq 1$ 2

Unit - V

9. (a) Show that l_0 is a maximal ideal of the Banach algebra l. 5

(b) If X is a Banach space, show that the Banach algebra $\mathbb{B}(X)$ is semisimple. 5

(c) Let A be a commutative Banach algebra. Let M_A be the set of all ideals of A and $£_A$ be the set of all complex homomorphisms on A. If $\phi(f) = \text{Ker } f \ \forall f \in £_A$, then show that $\phi : £_A \to M_A$ is a one-one, onto map. 4

10. (a) Show that every commulative Banach algebra which is not isometrically isomorphic to the Banach algebra $\mathbb{C}$ has a nonzero proper ideal. 4

(b) Let f be a complex homomorphism on a Banach algebra X. Show that 5

(i) f (e) = e

(ii) $f(x) \neq 0$ for all $x \in g(X)$

(iii) $|f(x)| < 1$ for all $x \in X$ with $||x|| < 1$.

(c) Let X be a compact Hausdorff space. Show that M is a maximal ideal of C(X) if and only if there exists $x \in X$ such that $M = \{ f \in C(X) : f(x) = 0 \}$ 5

★★★

PG Even Semester Examination, May 2025

MATHEMATICS

4th Semester

Course No. : MAT - 653 (B)

(Operator Theory)

Full Marks : 70

Pass Marks : 28

Time : 3 hours

The figures in the margin indicate full marks for the questions

(Answer one question from each unit)

Unit - I

1. (a) Show that every Hilbert space is reflexive. 6

(b) Let (x_n) and (y_n) be sequences in a Hilbert space H and let $x, y \in$ H. Show that if $x_n \to x$ and $y_n \to y$, then $<x_n, y_n> \to <x, y>$ 3

(c) Let H be a Hilbert space and let $T \in \mathbb{B}(H)$ be self-adjoint. Prove that 5

$$\|T\| = \sup \{ | \langle T_x, x \rangle | : x \in S_H \}$$

2. (a) Let H be a Hilbert space and let $T \in \mathbb{B}(H)$ be a normal operator. Show that $\| T^2 \| = \| T \|^2$ 2+4=6

(b) Show that the set of all s normal operators on a Hilbert space ^{H} is a closed subspace of $\mathbb{B}(H)$. 5

(c) If $P : \mathbb{C}^2 \to \mathbb{C}^2$ is given by $p(x, y) = (x - y, 0)$ for all $x, y \to \mathbb{C}$, show that P is a projection. Is it an orthogonal projection? Justify. 4

(Turn Over)

Unit - II

3. (a) Let X, Y be Banach spaces and let T : X→Y be an infinite rank compact operator. Show that T (X) is not closed in Y. 5

(b) Let X, Y be normed linear spaces and let (T_n) be a sequence of bounded linear operators in $\mathbb{B}$ (X, Y). Let T ∈ $\mathbb{B}$(X, Y).
Show that if $T_n \xrightarrow{SOT} T$, then $T_n \xrightarrow{WOT} T$. Is the converse tree? Justify. 2+3=5

(c) Let H, K be Hilbert spaces and let T ∈ $\mathbb{B}$(H, K). Show that T is compact iff T* is compact. 4

4. (a) Let X, Y be Banach spaces and let T ∈ $\mathbb{B}_0$(X, Y). Show that $\tilde{T} : X/Ker\ T \to Y$ defined by $\tilde{T}\ (x + Ker\ T) = T_x\ \forall\ x\ \in X$ is a compact opeartor. 4

(b) Let T : $l^2 \to l^2$ be given by T(x) = $(\alpha_n\ x_n)$ ∀ x = $(x_n) \in l^2$ where $(\alpha_n) \in l^\infty$. Show that T is compact if and only if $(\alpha_n) \in l_0$. Also, find T* : $l^2 \to l^2$. 5+2=7

(c) Let X and Y be normed linear spaces and let T : X→Y be a bounded linear operator. Show that if dim T(x)<∝, then T is compact. 3

Unit - III

5. (a) Let H be a non-zero complex Hilbert space and let T∈$\mathbb{B}$(H). Show that σ(T) is a non empty compact subset of $\mathbb{C}$. 7

(b) Let X be an infinite dimensional normed linear space and let T∈$\mathbb{B}$(X) be a finite rank operator. Show that
σ_{ap} (T) = σ_{cp} (T) = σ_p (T) = σ(T) 7

6. (a) Let T : $l^2 \to l^2$ be defined by T(x) = (0, x_1, x_2, ...) ∀x =$(x_n) \in l^2$. Find σ_p (T) , σ_r(T) and σ_c(T). 5

(b) Let H be a non zero complex Hilbert space and let T∈$\mathbb{B}$(H). Show that the spectral radius of T is given by $r_\sigma\ (T) = \lim_{n\to\infty} \|T^n\|^{1/n}$ 6

(c) Define numerical range of a bounded linear operator defind on a complex Hilbert space. Find the numerical range of $T : \mathbb{C}^2 \to \mathbb{C}^2$ defined by T(x, y) = (x, 0) ∀ x, y ∈ $\mathbb{C}$. 1+2=3

Unit - IV

7. (a) Let X be an algebra with the unit element e≠0 and let $(X, \|.\|)$ be a Banach space. If the multiplication is continuous co-ordinatewise, then show that there exists a norm $\|.\|_0$ on X such that $\|.\|_0$ is equivalent to $\|.\|$, $(X, \|.\|_0)$ is a Banach algebra and $\| e \|_0 = 1$. 7

(b) Let $(X, \|.\|)$ be a Unital Banach algebra and let x∈X. If $\| x \| < 1$, then show that 1-x is invertible in X. 3

(c) Define a topological zero divisor in a Banach algebra. Give an example, with proper justification of a topological zero divisor in a Banach algebra. 1+3=4

8. (a) Let X be a Banach algebra. Show that
Z (X) = $\{ x \in X : xy = yx\ \forall\ y \in x \}$ is a Banach sub algebra of X. 5

(b) Show that $GL_n(\mathbb{C})$ is dense in the Banach algebra $M_n(\mathbb{C})$. Show also that g(l^2) is not dense in the Banach algebra $\mathbb{B}(l^2)$ 4+3=7

(Turn Over)