Unit - V

9. (a) Let H and K be Hilbert spaces and let $T \in \mathbb{B}(H, K)$. Show that there exists unique $T^* \in \mathbb{B}(K, H)$ such that $\langle Tx, y \rangle = \langle x, T^*y \rangle \; \forall \; x \in H, \; \forall \; y \in K$ show also that $||T|| = ||T^*||$ 8

(b) Define a compact operator. Show that if X and Y are normed linear spaces and if $\mathbb{B}(X, Y)$, $\mathbb{B}_0(X, Y)$, $\mathbb{B}_{00}(X, Y)$ represent the set of all bounded linear, compact, finite rank bounded linear maps respectively from X to Y, then 1+5=6

$$\mathbb{B}_{00}(X, Y) \subset B_0(X, Y) \subset \mathbb{B}(X, Y)$$

10. (a) Let H and K be Hilbert spaces and let $T \in \mathbb{B}(H, K)$. Show that 4+4=8

$$\text{Ker } T = \overline{\text{range } (T^*)}^{\perp}$$

$$\text{Ker } T^{\perp} = \overline{\text{range } (T^*)}$$

(b) Show that

$T : l^2 \to l^2$

defined by

$T(x) = (\alpha_n x_n) \; \forall \; x = (x_n) \in l^2$ is a compact linear operator if $\alpha_n \to 0$ 6

★★★

PG (CBCS) Even Semester Examination, May 2025

MATHEMATICS

4th Semester

Course No. : MTMCC - 401 (O)

(Functional Analysis)

Full Marks : 70

Pass Marks : 28

Time : 3 hours

The figures in the margin indicate full marks for the questions

(Answer one question from each unit)

Unit - I

1. (a) Define a normed linear space. Let $(X, \|.\|)$ be a normed linear space. Show that $d : X \times X \to \mathbb{R}$ defined by $d(x, y) = \|x - y\|$ for all $x, y \in X$ is a metric on X. 2+5=7

(b) State Riesz lemma. Show that a normed linear space X if finite dimensional if and only if $B_X = \{ x \in X : ||x|| \leq 1 \}$ is compact. 1+6=7

2. (a) Let $X \neq \{0\}$, Y be normed linear spaces. Show that $\forall \; T \in \mathbb{B}(X, Y)$ 7

$$\|T\| = \sup \{ x \in X : \|x\| = 1 \}$$

$$= \sup \left\{ \frac{\|T_x\|}{\|x\|} : x \neq 0 \; in \; X \right\}$$

$$= \inf \{ M > 0 : \|T_x\| \leq M \|x\| \}$$

(b) Define a Banach space. Show that $(l^{\infty}, ||.||_{\infty})$ is a Banach space, where $l^{\infty} = \{ (x_n) \in \mathbb{K}^{\mathbb{N}} : (x_n)$ is bounded $\}$ and
$||x||_{\infty} = \sup \{ |x_n| \} : n \in \mathbb{N} \}$
$\forall\ x = (x_n) \in l^{\infty}$ 1+6=7

Unit - II

3. (a) Let X be a normed linear space, Y a Banach space and let Z be a dense subspace of X. Then show that for every $S \in \mathbb{B}(Z, Y)$, there exists unique $T \in \mathbb{B}(X, Y)$ such that $T|_Z = S$ and $||T|| = ||S||$ 7

(b) State Hahn-Banach theorem. Using Hahn-Banach theorem. Show that if X is a normed linear space, Y a closed subspace of X and $x_0 \in X - Y$, then there exists $f \in X^*$ such that $f(x_0) = \text{dist}(x_0, Y)$ and $Y \subset \ker f$. 1+6=7

4. (a) Show that the topological dual of $(\mathbb{K}^n, ||.||_1)$ is isometrically isomorphic to $(\mathbb{K}^n, ||.||_{\infty})$ 7

(b) Define a reflexive space. Show that if X is a Banach space, then X is reflexive iff X^* is reflexive. 1+6=7

Unit - III

5. (a) State and prove uniform boundedness principle. 1+6=7

(b) State and prove open mapping theorem. 1+6=7

6. (a) State and prove Banach-Steinhauss theorem. 1+4=5

(b) State and prove closed graph theorem. 1+5=6

(c) Give an example of a linear map $T : X \to Y$ whose graph is not closed in XxY, with justification. 3

Unit - IV

7. (a) Define a Hilbert space. Let H be a Hilbert space and let C be a closed convex nonempty subset of H. If $x_0 \in H - C$, show that there exists unique $y_0 \in C$ such that
$||x_0 - y_0|| = \text{dist}(x_0, C)$ 1+5=6

(b) Let X be an inner product space and let $x_0 \in X$. Then show that $f : X \to \mathbb{K}$ given by $f(x) = \langle x, x_0 \rangle\ \forall\ x \in X$ is a bounded linear functional with $||f|| = ||x_0||$ 4

(c) Let H be a Hilbert space and let M be a closed subspace of H. Show that $M^{\perp\perp} = M$ 4

8. (a) Let X be an inner product space and let A be a non empty subset of X. Show that 2+5=7
(i) $A \cap A^{\perp} = \{0\}$ if $0 \in A$
(ii) $A^{\perp} = \{ x \in X : \langle x, a \rangle = 0\ \forall\ a \in A \}$
is a closed subspace of X.

(b) State and prove Riesz representation theorem. 1+6=7