(b) Let f be integrable over $\mathbb{E}$. Let A and B are disjoint measurable subsets of $\mathbb{E}$. Show that

$$\int_{A\cup B} f = \int_A f + \int_B f$$

Unit - V

9. (a) Let f be integrable over E and $<E_n>$ be a sequence of disjoint measurable subsets of E whose union is E. Show that $\int_E f = \sum_{n=1}^{\infty} \int_{En} f$ 7

(b) Assume E has finite measure. Let the sequence of functions $<f_n>$ be uniformly integrable over $\mathbb{E}$. If $f_n \to f$ pointwise i.e. on $\mathbb{E}$, show that f is integrable over $\mathbb{E}$. 7

10. (a) State and prove Vitali's convergence theorem. 7

(b) Let $<f_n>$ be a sequence of functions on $\mathbb{E}$ that is uniformly integrable and tight over $\mathbb{E}$. Suppose $f_n \to f$ pointwise a.e. on $\mathbb{E}$. Show that f is integrable over $\mathbb{E}$ and 7

$$\lim_{n\to\infty} \int_{\mathbb{E}} f_n = \int_{\mathbb{E}} f$$

★★★

PG (CBCS) Even Semester Examination, May 2025

MATHEMATICS

4th Semester

Course No. : MTMCC - 402 (O)

(Real Analysis II)

Full Marks : 70

Pass Marks : 28

Time : 3 hours

The figures in the margin indicate full marks for the questions

(Answer one question from each unit)

Unit - I

1. (a) Let $<f_n>$ be a sequence of functions defined on [a, b] such that $f_n \to f$ uniformly. Show that f is continuous in [a, b] 5

(b) Let $f_n : \mathbb{R} \to \mathbb{R}$, $f_n(x) := \sqrt{x^2 + 1/n}$ for each $x \in \mathbb{R}$ and $n \geq 1$. Show that $f_n \to f$ uniformly in $\mathbb{R}$ where $f(x) = |x| \ \forall\ x \in \mathbb{R}$. 5

(c) Let $f : |a, b| \to \mathbb{R}$ be bounded and monotonic. Show that f is Riemann integrable. 4

2. (a) Let $<f_n>$ be a sequence of Riemann integrable

functions in [a, b] such that $f_n \to f$ uniformly in [a, b]. Show that f is also Riemann integrable in [a, b] and $\lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b f\,dx$ 6

(b) Test for pointwise and uniform convergence of the sequence $< f_n >$ where $f_n(x) := x^n \ \forall\ x \in [0, 1]$ and $\forall\ n \geq 1$ 4

(c) Show that every power series represents a differentiable function within its domain of convergence. 4

Unit - II

3. (a) Show that outer measure of every interval is its length. 7

(b) Show that the set of all Lebesgue measurable sets form a σ-algebra. 7

4. (a) Show that if f and g are measurable over a measurable domain D, then $f+g$ and fg are also measurable. 7

(b) Show that every Borel subset of $\mathbb{R}$ is Lebesgue measurable. 7

Unit - III

5. (a) Show that every non-negative measurable function on a measurable domain E is the

pointwise limit of a sequence of simple functions in $\mathbb{E}$. 7

(b) State and prove Lebesgue monotone convergence theorem. 1+6=7

6. (a) State and prove Fatou's lemma. 1+6=7

(b) Let φ and ψ be simple measurable functions in a measurable set $\mathbb{E}$ with $m(\mathbb{E}) < \infty$. Show that 7

$$\int_{\mathbb{E}} \alpha\varphi + \beta\psi = \alpha \int_{\mathbb{E}} \varphi + \beta \int_{\mathbb{E}} \psi$$

Unit - IV

7. (a) State and prove Lebesgue dominated convergence theorem. 1+6=7

(b) Let f be bounded and Riemann integrable over [a, b]. Show that f is also Lebesgue integrable over [a, b] and 7

$$\int_{[a,b]} f = \int_a^b f(x)\,dx$$

8. (a) Let f be a measurable function in a measurable set $\mathbb{E}$. Show that $|f|$ is integrable over $\mathbb{E}$ iff f^+ and f^- are both integrable over $\mathbb{E}$. 7